Computer Algebra
Independent Integration Tests

Summer 2023 edition with Rubi V 4.17.3

4-Trig-functions/4.6-Cosecant/129-4.6.1.2-d-csc-^n-a+b-csc-^m

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Contents

CHAPTER 1

Introduction

This report gives the result of running the computer algebra independent integration test.The download section in on the main webpage contains links to download the problems in plain text format used for all CAS systems. The number of integrals in this report is [59]. This is test number [129].

1.1 Listing of CAS systems tested

The following are the CAS systems tested:

1. Mathematica 13.3.1 (August 16, 2023) on windows 10.
2. Rubi 4.17.3 (Sept 25, 2023) on Mathematica 13.3.1 on windows 10
3. Maple 2023.1 (July, 12, 2023) on windows 10.
4. Maxima 5.47 (June 1, 2023) using Lisp SBCL 2.3.0 on Linux via sagemath 10.1 (Aug 20, 2023).
5. FriCAS 1.3.9 (July 8, 2023) based on sbcl 2.3.0 on Linux via sagemath 10.1 (Aug 20, 2023).
6. Giac/Xcas 1.9.0-57 (June 26, 2023) on Linux via sagemath 10.1 (Aug 20, 2023).
7. Sympy 1.12 (May 10, 2023) Using Python 3.11.3 on Linux.
8. Mupad using Matlab 2021a with Symbolic Math Toolbox Version 8.7 on windows 10.

Maxima and Fricas and Giac are called using Sagemath. This was done using Sagemath `integrate` command by changing the name of the algorithm to use the different CAS systems.

Sympy was run directly in Python not via sagemath.

1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or `Hypergeometric2F1` functions. `RootSum` and `RootOf` are not allowed. If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

System	% solved	% Failed
Rubi	100.00 (59)	0.00 (0)
Mathematica	89.83 (53)	10.17 (6)
Maple	69.49 (41)	30.51 (18)
Fricas	69.49 (41)	30.51 (18)
Giac	67.80 (40)	32.20 (19)
Mupad	55.93 (33)	44.07 (26)
Maxima	42.37 (25)	57.63 (34)
Sympy	5.08 (3)	94.92 (56)

Table 1.1: Percentage solved for each CAS

The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

grade	description
A	Integral was solved and antiderivative is optimal in quality and leaf size.
B	Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size.
C	Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons 1. antiderivative contains a hypergeometric function and the optimal antiderivative does not. 2. antiderivative contains a special function and the optimal antiderivative does not. 3. antiderivative contains the imaginary unit and the optimal antiderivative does not.
F	Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised.

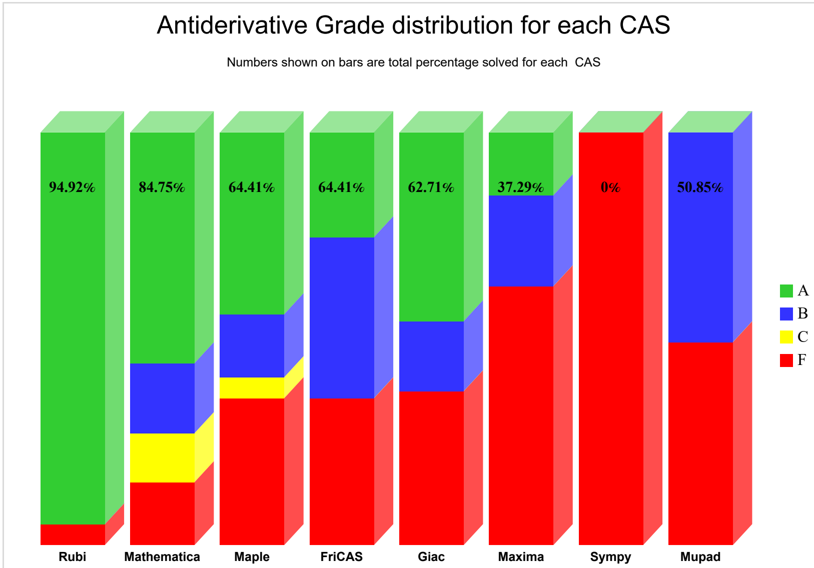
Table 1.2: Description of grading applied to integration result

Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

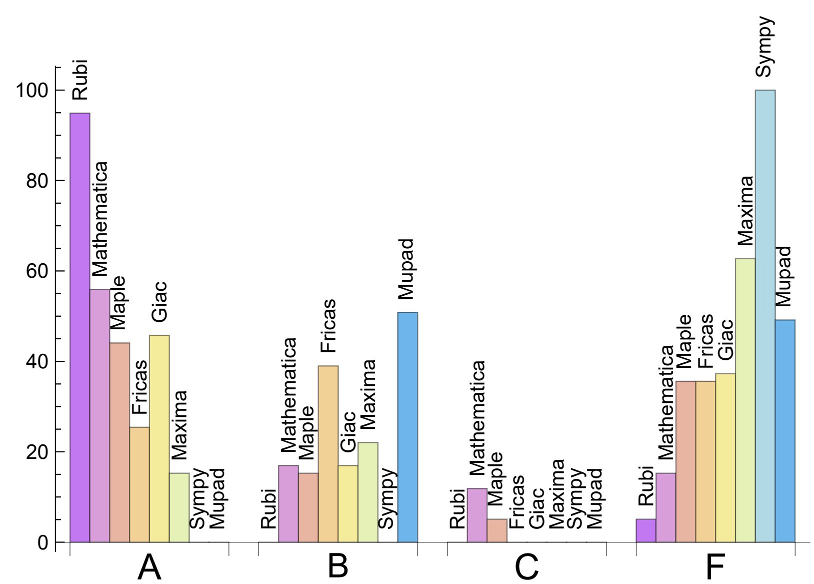
System	% A grade	% B grade	% C grade	% F grade
Rubi	94.915	0.000	0.000	5.085
Mathematica	55.932	16.949	11.864	15.254
Giac	45.763	16.949	0.000	37.288
Maple	44.068	15.254	5.085	35.593
Fricas	25.424	38.983	0.000	35.593
Maxima	15.254	22.034	0.000	62.712
Mupad	0.000	50.847	0.000	49.153
Sympy	0.000	0.000	0.000	100.000

Table 1.3: Antiderivative Grade distribution of each CAS

The following is a Bar chart illustration of the data in the above table.



The figure below compares the grades of the CAS systems.



The following table shows the distribution of the different types of failures for each CAS. There are 3 types failures. The first is when CAS returns the input within the time limit, which means it could not solve it. This is the typical failure and given as **F**.

The second failure is due to time out. CAS could not solve the integral within the 3 minutes time limit which is assigned. This is assigned **F(-1)**.

The third is due to an exception generated, indicated as **F(-2)**. This most likely indicates an interface problem between sagemath and the CAS (applicable only to FriCAS, Maxima and Giac) or it could be an indication of an internal error in the CAS itself. This type of error requires more investigation to determine the cause.

System	Number failed	Percentage normal failure	Percentage time-out failure	Percentage exception failure
Rubi	0	0.00	0.00	0.00
Mathematica	6	100.00	0.00	0.00
Fricas	18	100.00	0.00	0.00
Maple	18	100.00	0.00	0.00
Giac	19	94.74	0.00	5.26
Mupad	26	0.00	100.00	0.00
Maxima	34	61.76	0.00	38.24
Sympy	56	94.64	5.36	0.00

Table 1.4: Failure statistics for each CAS

1.3 Time and leaf size Performance

The table below summarizes the performance of each CAS system in terms of time used and leaf size of results.

Mean size is the average leaf size produced by the CAS (before any normalization). The Normalized mean is relative to the mean size of the optimal anti-derivative given in the input files.

For example, if CAS has **Normalized mean** of 3, then the mean size of its leaf size is 3 times as large as the mean size of the optimal leaf size.

Median size is value of leaf size where half the values are larger than this and half are smaller (before any normalization). i.e. The Middle value.

Similarly the **Normalized median** is relative to the median leaf size of the optimal.

For example, if a CAS has Normalized median of 1.2, then its median is 1.2 as large as the median leaf size of the optimal.

System	Mean time (sec)
Fricas	0.27
Giac	0.31
Rubi	0.43
Maxima	0.46
Maple	0.70
Mathematica	2.96
Sympy	8.74
Mupad	19.44

Table 1.5: Time performance for each CAS

System	Mean size	Normalized mean	Median size	Normalized median
Sympy	17.00	0.98	19.00	1.00
Mathematica	96.36	1.35	76.00	1.17
Maxima	112.28	2.38	95.00	1.85
Rubi	120.86	1.07	74.00	1.00
Giac	127.35	2.04	88.50	1.58
Maple	141.07	2.01	73.00	1.23
Fricas	253.59	3.42	181.00	3.05
Mupad	744.76	5.87	89.00	1.81

Table 1.6: Leaf size performance for each CAS

1.4 Performance based on number of rules Rubi used

This section shows how each CAS performed based on the number of rules Rubi needed to solve the same integral. One diagram is given for each CAS.

On the y axis is the percentage solved which Rubi itself needed the number of rules given the x axis. These plots show that as more rules are needed then most CAS system percentage of solving decreases which indicates the integral is becoming more complicated to solve.

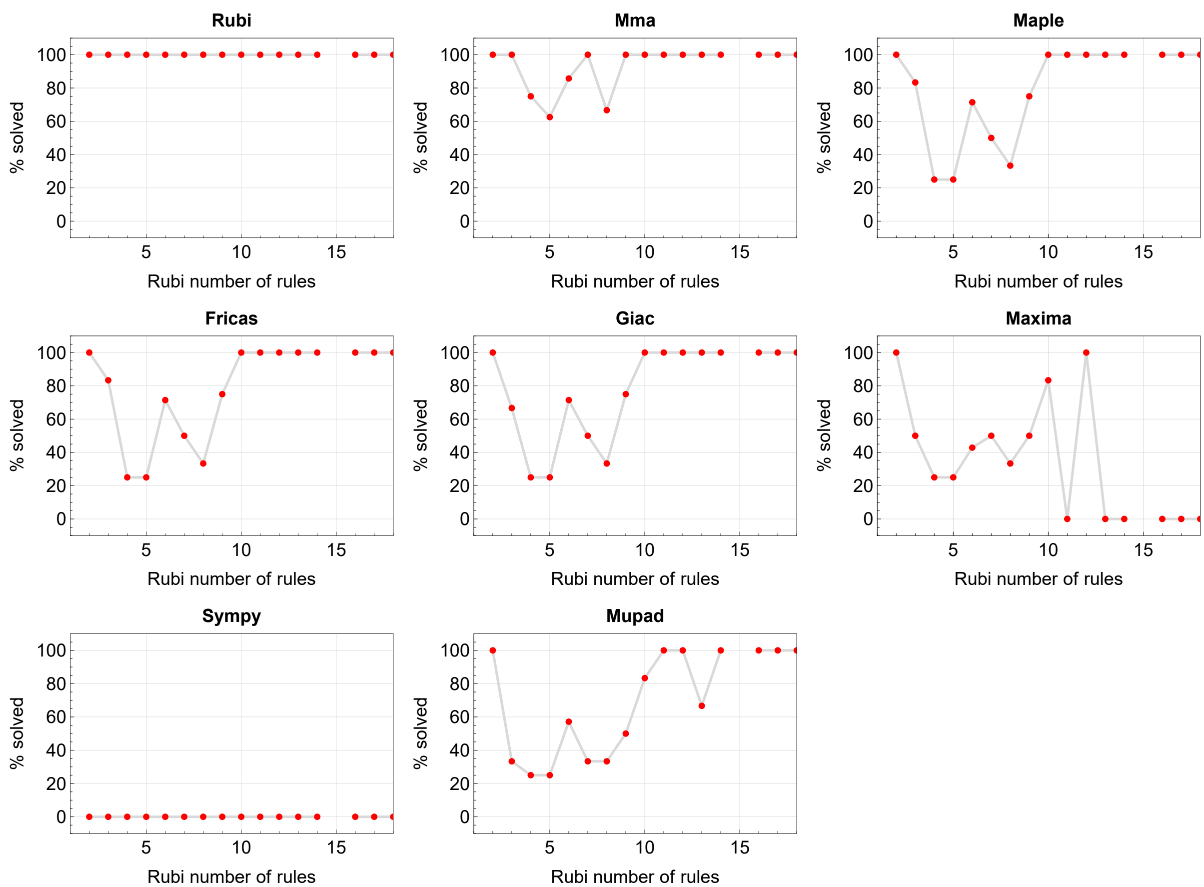


Figure 1.1: Solving statistics per number of Rubi rules used

1.5 Performance based on number of steps Rubi used

This section shows how each CAS performed based on the number of steps Rubi needed to solve the same integral. Note that the number of steps Rubi needed can be much higher than the number of rules, as the same rule could be used more than once.

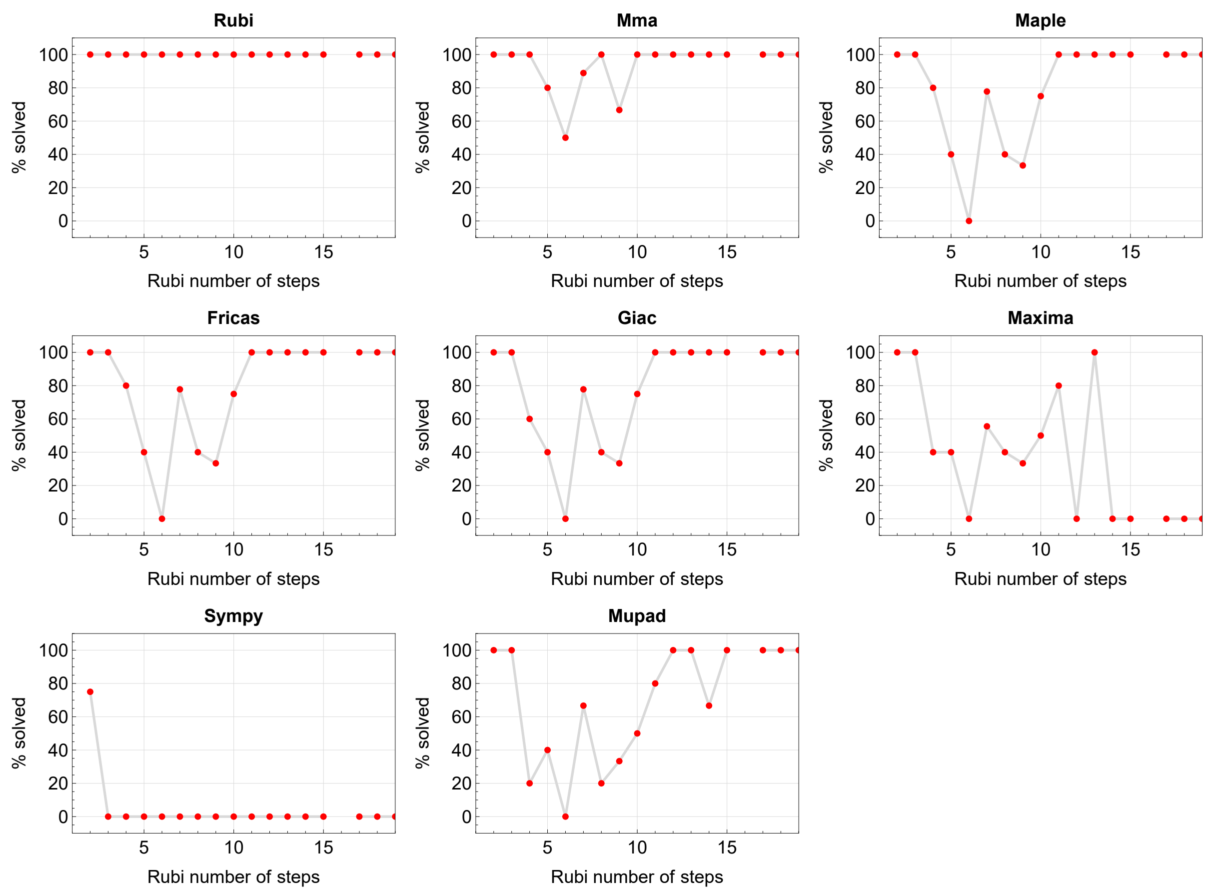


Figure 1.2: Solving statistics per number of Rubi steps used

The above diagram show that the precentage of solved intergals decreases for most CAS systems as the number of steps increases. As expected, for integrals that required less steps by Rubi, CAS systems had more success which indicates the integral was not as hard to solve. As Rubi needed more steps to solve the integral, the solved percentage decreased for most CAS systems which indicates the integral is becoming harder to solve.

1.6 Solved integrals histogram based on leaf size of result

The following shows the distribution of solved integrals for each CAS system based on leaf size of the antiderivatives produced by each CAS. It shows that most integrals solved produced leaf size less than about 100 to 150. The bin size used is 40.

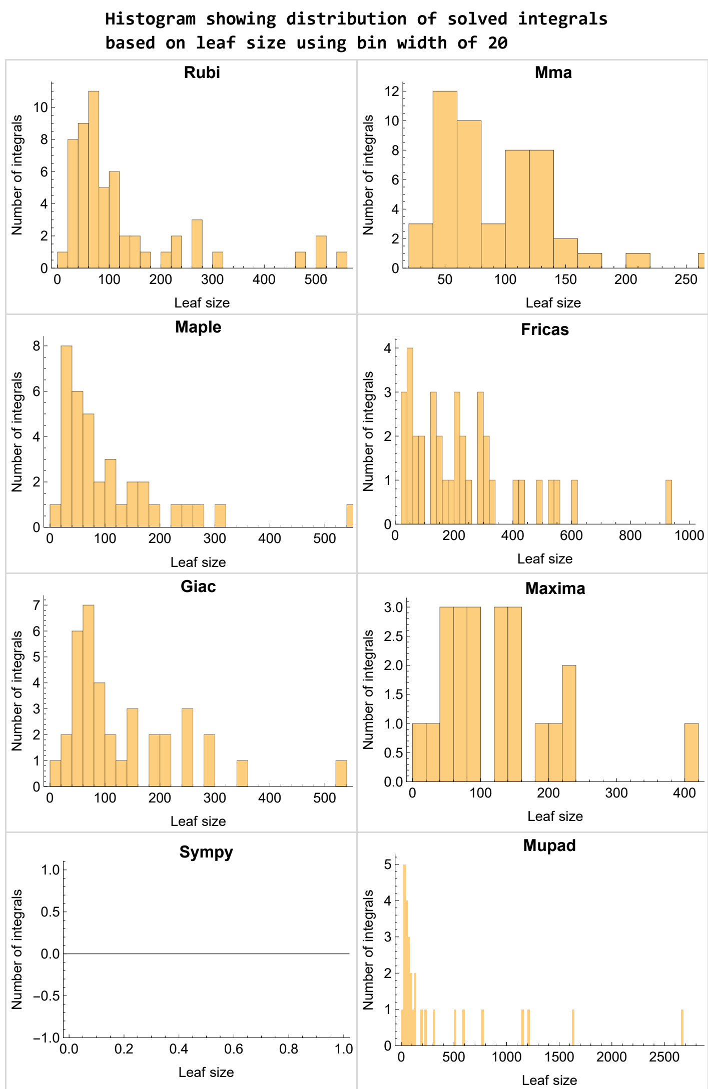


Figure 1.3: Solved integrals based on leaf size distribution

1.7 Solved integrals histogram based on CPU time used

The following shows the distribution of solved integrals for each CAS system based on CPU time used in seconds. The bin size used is 0.1 second.

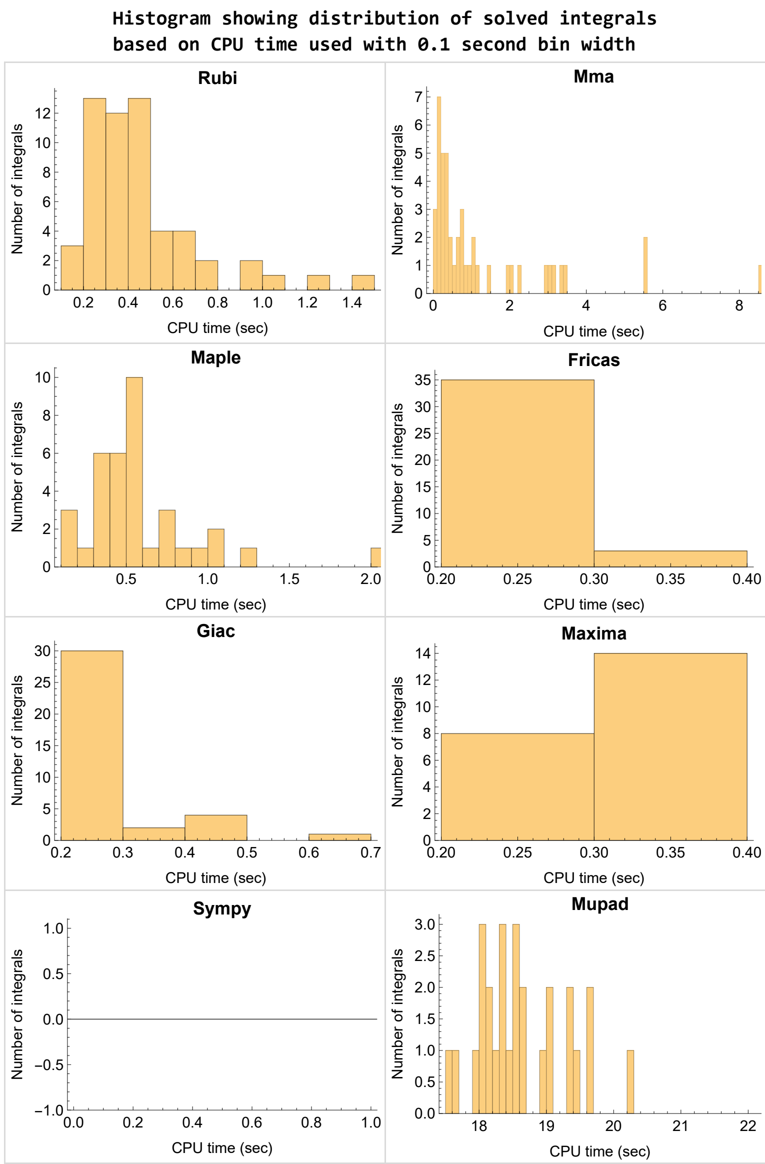


Figure 1.4: Solved integrals histogram based on CPU time used

1.8 Leaf size vs. CPU time used

The following gives the relation between the CPU time used to solve an integral and the leaf size of the antiderivative.

The result for Fricas, Maxima and Giac is shifted more to the right than the other CAS system due to the use of sagemath to call them, which causes an initial slight delay in the timing to start the integration due to overhead of starting a new process each time.

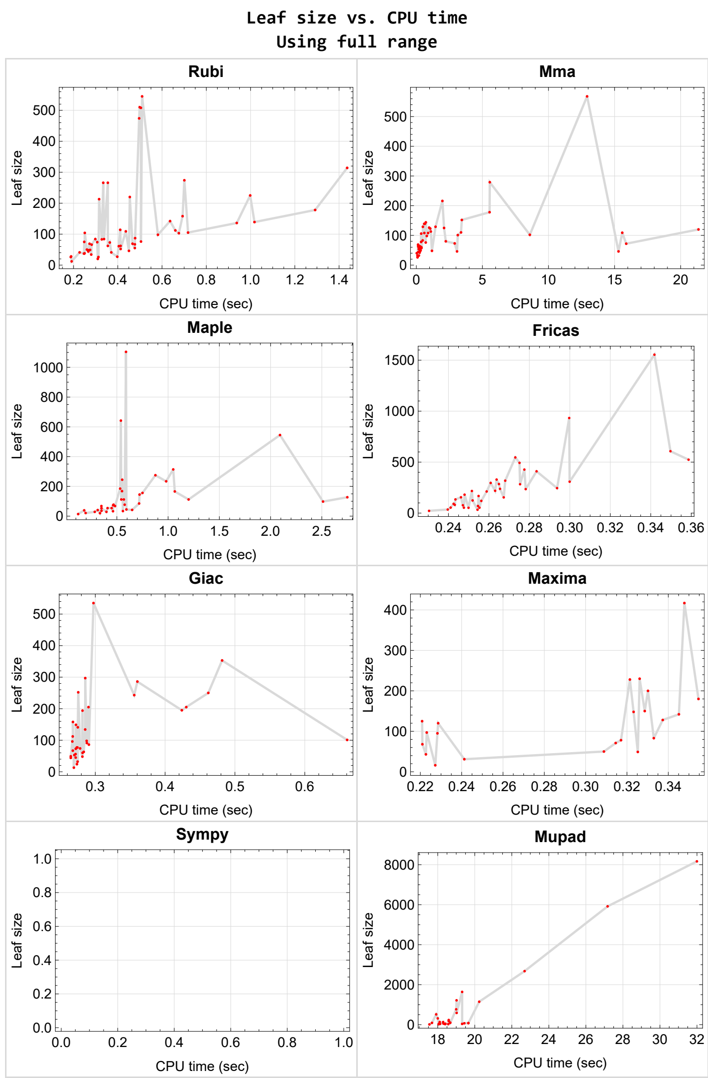


Figure 1.5: Leaf size vs. CPU time. Full range

1.9 list of integrals with no known antiderivative

$\{57, 58, 59\}$

1.10 List of integrals solved by CAS but has no known antiderivative

Rubi $\{\}$

Mathematica $\{\}$

Maple $\{\}$

Maxima $\{\}$

Fricas $\{\}$

Sympy $\{\}$

Giac $\{\}$

Mupad $\{\}$

1.11 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not necessarily mean that the anti-derivative is wrong as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it possible to do further investigation to determine why the result could not be verified.

Rubi $\{24, 25, 26, 27\}$

Mathematica $\{\}$

Maple $\{14, 17, 18\}$

Maxima Verification phase not currently implemented.

Fricas Verification phase not currently implemented.

Sympy Verification phase not currently implemented.

Giac Verification phase not currently implemented.

Mupad Verification phase not currently implemented.

1.12 Timing

The command `AbsoluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of_int',int(expr,x)),output='realtime'
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 CPU minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out was not counted in the final statistics.

1.13 Verification

A verification phase was applied on the result of integration for `Rubi` and `Mathematica`.

Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative was correct.

Verification phase also had 3 minutes time out. An integral whose result was not verified could still be correct, but further investigation is needed on those integrals. These integrals were marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

1.14 Important notes about some of the results

1.14.1 Important note about Maxima results

Since tests were run in a batch mode, and using an automated script, then any integral where Maxima needed an interactive response from the user to answer a question during the evaluation of the integral will fail.

The exception raised is `ValueError`. Therefore Maxima results is lower than what would result if Maxima was run directly and each question was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the `Timofeev` test file, there were about 14 such integrals out of total 705, or about 2 percent. This percentage can be higher or lower depending on the specific input test file.

Such integrals can be identified by looking at the output of the integration in each section for Maxima. The exception message will indicate the cause of error.

Maxima integrate was run using SageMath with the following settings set by default

```
'besselexpand : true'
'display2d : false'
'domain : complex'
'keepfloat : true'
'load(to_poly_solve)'
'load(simplify_sum)'
'load(abs_integrate)' 'load(diag)'
```

SageMath automatic loading of Maxima `abs_integrate` was found to cause some problems. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib
maxima_lib.set('extra_definite_integration_methods', '[]')
maxima_lib.set('extra_integration_methods', '[]')
```

See `https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/` for reference.

1.14.2 Important note about FriCAS result

There were few integrals which failed due to SageMath interface and not because FriCAS system could not do the integration.

These will fail With error `Exception raised: NotImplementedError`.

The number of such cases seems to be very small. About 1 or 2 percent of all integrals. These can be identified by looking at the exception message given in the result.

1.14.3 Important note about finding leaf size of antiderivative

For Mathematica, Rubi, and Maple, the builtin system function `LeafSize` was used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special builtin function for this purpose at this time. Therefore the leaf size for Fricas and Sympy antiderivative was determined using the following function, thanks to user `slelievre` at `https://ask.sagemath.org/question/57123/could-we-have-a-leaf_count-function-in-base-sagemath/`

```
def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)
```

For Sympy, which was called directly from Python, the following code was used to obtain the leafsize of its result

```
try:
  # 1.7 is a fudge factor since it is low side from actual leaf count
  leafCount = round(1.7*count_ops(anti))

  except Exception as ee:
        leafCount =1
```

1.14.4 Important note about Mupad results

Matlab's symbolic toolbox does not have a leaf count function to measure the size of the antiderivative. Maple was used to determine the leaf size of Mupad output by post processing Mupad result.

Currently no grading of the antiderivative for Mupad is implemented. If it can integrate the problem, it was assigned a B grade automatically as a placeholder. In the future, when grading function is implemented for Mupad, the tests will be rerun again.

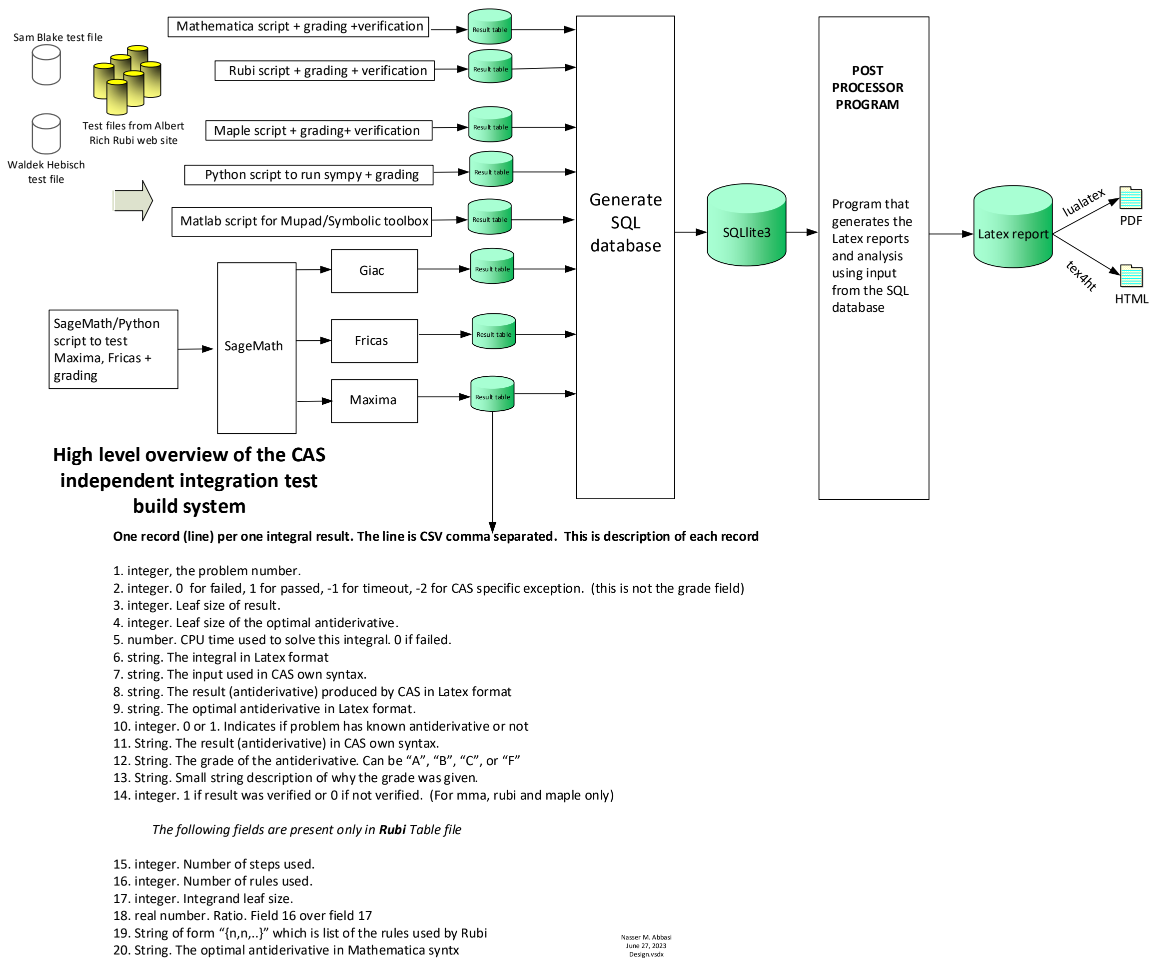
The following is an example of using Matlab's symbolic toolbox (Mupad) to solve an integral

```
integrand = evalin(symengine,'cos(x)*sin(x)')
the_variable = evalin(symengine,'x')
anti = int(integrand,the_variable)
```

Which gives `sin(x)^2/2`

1.15 Design of the test system

The following diagram gives a high level view of the current test build system.



CHAPTER 2

DETAILED SUMMARY TABLES OF RESULTS

2.1 List of integrals sorted by grade for each CAS

2.1.1 Rubi

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56 }

B grade { }

C grade { }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.2 Mma

A grade { 2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 28, 29, 30, 31, 32, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 53 }

B grade { 1, 3, 4, 5, 19, 20, 36, 37, 38, 52 }

C grade { 21, 22, 23, 24, 25, 26, 27 }

F normal fail { 33, 34, 35, 54, 55, 56 }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.3 Maple

A grade { 1, 2, 3, 4, 5, 7, 8, 9, 10, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 52, 53 }

B grade { 13, 14, 15, 16, 17, 18, 19, 20, 51 }

C grade { 6, 11, 12 }

F normal fail { 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 54, 55, 56 }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.4 Fricas

A grade { 5, 6, 7, 8, 9, 10, 16, 43, 44, 45, 46, 47, 48, 52, 53 }

B grade { 1, 2, 3, 4, 11, 12, 13, 14, 15, 17, 18, 19, 20, 36, 37, 38, 39, 40, 41, 42, 49, 50, 51 }

C grade { }

F normal fail { 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 54, 55, 56 }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.5 Maxima

A grade { 4, 5, 6, 16, 36, 37, 38, 52, 53 }

B grade { 1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17 }

C grade { }

F normal fail { 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 54, 55, 56 }

F(-1) timedout fail { }

F(-2) exception fail { 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 }

2.1.6 Giac

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 37, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 52, 53 }

B grade { 13, 14, 15, 16, 17, 18, 20, 36, 38, 51 }

C grade { }

F normal fail { 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 54, 55, 56 }

F(-1) timedout fail { }

F(-2) exception fail { 19 }

2.1.7 Mupad

A grade { }

B grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53 }

C grade { }

F normal fail { }

F(-1) timedout fail { 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 54, 55, 56 }

F(-2) exception fail { }

2.1.8 Sympy

A grade { }

B grade { }

C grade { }

F normal fail { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56 }

F(-1) timedout fail { 21, 24, 47 }

F(-2) exception fail { }

2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by the table below. The elapsed time is in seconds. For failed result it is given as **F(-1)** if the failure was due to timeout. It is given as **F(-2)** if the failure was due to an exception being raised, which could indicate a bug in the system. If the failure was due to integral not being evaluated within the time limit, then it is given as **F**.

In this table, the column `N.S.` means **normalized size** and is defined as $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$. To make the table fit the page, the name `Mathematica` was abbreviated to `MMA`.

Problem 1	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	A	B	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	55	55	113	68	120	168	0	96	89
N.S.	1	1.00	2.05	1.24	2.18	3.05	0.00	1.75	1.62
time (sec)	N/A	0.450	1.095	0.482	0.229	0.255	0.000	0.267	17.689

Problem 2	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	B	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	44	46	83	54	97	134	0	73	69
N.S.	1	1.05	1.89	1.23	2.20	3.05	0.00	1.66	1.57
time (sec)	N/A	0.432	0.481	0.409	0.223	0.243	0.000	0.272	19.437

Problem 3	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	A	B	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	27	27	63	36	68	91	0	53	49
N.S.	1	1.00	2.33	1.33	2.52	3.37	0.00	1.96	1.81
time (sec)	N/A	0.380	0.290	0.350	0.221	0.243	0.000	0.270	18.414

Problem 4	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	A	A	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	20	20	44	21	31	53	0	24	23
N.S.	1	1.00	2.20	1.05	1.55	2.65	0.00	1.20	1.15
time (sec)	N/A	0.293	0.143	0.195	0.241	0.248	0.000	0.274	18.045

Problem 5	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	12	12	26	14	16	22	0	13	13
N.S.	1	1.00	2.17	1.17	1.33	1.83	0.00	1.08	1.08
time (sec)	N/A	0.179	0.089	0.122	0.227	0.230	0.000	0.269	17.550

Problem 6	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	28	28	47	29	50	54	0	32	27
N.S.	1	1.00	1.68	1.04	1.79	1.93	0.00	1.14	0.96
time (sec)	N/A	0.182	0.261	0.284	0.309	0.241	0.000	0.275	18.400

Problem 7	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	B	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	25	26	32	20	78	35	0	44	46
N.S.	1	1.04	1.28	0.80	3.12	1.40	0.00	1.76	1.84
time (sec)	N/A	0.304	0.190	0.335	0.317	0.240	0.000	0.265	18.344

Problem 8	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	B	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	40	41	42	29	128	53	0	56	59
N.S.	1	1.02	1.05	0.72	3.20	1.32	0.00	1.40	1.48
time (sec)	N/A	0.352	0.252	0.398	0.337	0.255	0.000	0.272	18.303

Problem 9	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	B	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	53	52	49	33	180	70	0	67	78
N.S.	1	0.98	0.92	0.62	3.40	1.32	0.00	1.26	1.47
time (sec)	N/A	0.388	0.323	0.462	0.355	0.255	0.000	0.268	19.655

Problem 10	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	B	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	66	67	57	45	230	81	0	91	93
N.S.	1	1.02	0.86	0.68	3.48	1.23	0.00	1.38	1.41
time (sec)	N/A	0.448	0.385	0.593	0.326	0.243	0.000	0.288	18.678

Problem 11	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	B	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	57	61	108	54	142	124	0	60	52
N.S.	1	1.07	1.89	0.95	2.49	2.18	0.00	1.05	0.91
time (sec)	N/A	0.389	0.602	0.449	0.345	0.252	0.000	0.282	18.078

Problem 12	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	B	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	88	98	123	77	228	181	0	86	78
N.S.	1	1.11	1.40	0.88	2.59	2.06	0.00	0.98	0.89
time (sec)	N/A	0.557	1.052	0.576	0.321	0.248	0.000	0.291	19.655

Problem 13	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	B	B	F	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	65	69	80	275	417	318	0	250	0
N.S.	1	1.06	1.23	4.23	6.42	4.89	0.00	3.85	0.00
time (sec)	N/A	0.447	2.229	0.876	0.348	0.268	0.000	0.462	0.000

Problem 14	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	B	B	F	B	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD
size	44	44	69	245	200	212	0	195	0
N.S.	1	1.00	1.57	5.57	4.55	4.82	0.00	4.43	0.00
time (sec)	N/A	0.257	0.130	0.552	0.330	0.259	0.000	0.424	0.000

Problem 15	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	B	B	F	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	26	26	32	166	148	120	0	353	0
N.S.	1	1.00	1.23	6.38	5.69	4.62	0.00	13.58	0.00
time (sec)	N/A	0.175	0.085	1.065	0.323	0.256	0.000	0.482	0.000

Problem 16	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	A	A	F	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	62	62	54	185	83	219	0	205	0
N.S.	1	1.00	0.87	2.98	1.34	3.53	0.00	3.31	0.00
time (sec)	N/A	0.328	0.179	0.532	0.333	0.263	0.000	0.430	0.000

Problem 17	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	B	B	F	B	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD
size	81	87	129	642	150	427	0	243	0
N.S.	1	1.07	1.59	7.93	1.85	5.27	0.00	3.00	0.00
time (sec)	N/A	0.460	0.495	0.539	0.329	0.278	0.000	0.356	0.000

Problem 18	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	F	B	F	B	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD
size	100	112	139	1104	0	546	0	286	0
N.S.	1	1.12	1.39	11.04	0.00	5.46	0.00	2.86	0.00
time (sec)	N/A	0.645	0.596	0.589	0.000	0.273	0.000	0.360	0.000

Problem 19	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	B	F	B	F	F(-2)	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	37	37	108	98	0	283	0	0	0
N.S.	1	1.00	2.92	2.65	0.00	7.65	0.00	0.00	0.00
time (sec)	N/A	0.238	0.887	2.512	0.000	0.275	0.000	0.000	0.000

Problem 20	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	B	F	B	F	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	38	38	101	127	0	296	0	101	0
N.S.	1	1.00	2.66	3.34	0.00	7.79	0.00	2.66	0.00
time (sec)	N/A	0.237	3.136	2.749	0.000	0.261	0.000	0.661	0.000

Problem 21	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	F	F	F	F(-1)	F	F(-1)
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	254	266	102	0	0	0	0	0	0
N.S.	1	1.05	0.40	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.345	8.587	0.000	0.000	0.000	0.000	0.000	0.000

Problem 22	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	213	213	46	0	0	0	0	0	0
N.S.	1	1.00	0.22	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.303	3.078	0.000	0.000	0.000	0.000	0.000	0.000

Problem 23	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	254	266	110	0	0	0	0	0	0
N.S.	1	1.05	0.43	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.323	3.380	0.000	0.000	0.000	0.000	0.000	0.000

Problem 24	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	F	F	F	F(-1)	F	F(-1)
verified	N/A	No	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	514	510	120	0	0	0	0	0	0
N.S.	1	0.99	0.23	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.477	21.349	0.000	0.000	0.000	0.000	0.000	0.000

Problem 25	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	F	F	F	F	F	F(-1)
verified	N/A	No	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	470	474	109	0	0	0	0	0	0
N.S.	1	1.01	0.23	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.475	15.581	0.000	0.000	0.000	0.000	0.000	0.000

Problem 26	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	F	F	F	F	F	F(-1)
verified	N/A	No	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	508	508	46	0	0	0	0	0	0
N.S.	1	1.00	0.09	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.479	15.297	0.000	0.000	0.000	0.000	0.000	0.000

Problem 27	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	F	F	F	F	F	F(-1)
verified	N/A	No	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	552	545	72	0	0	0	0	0	0
N.S.	1	0.99	0.13	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.498	15.881	0.000	0.000	0.000	0.000	0.000	0.000

Problem 28	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	48	48	48	0	0	0	0	0	0
N.S.	1	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.257	1.180	0.000	0.000	0.000	0.000	0.000	0.000

Problem 29	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	69	69	73	0	0	0	0	0	0
N.S.	1	1.00	1.06	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.260	2.903	0.000	0.000	0.000	0.000	0.000	0.000

Problem 30	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	156	158	178	0	0	0	0	0	0
N.S.	1	1.01	1.14	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.669	5.541	0.000	0.000	0.000	0.000	0.000	0.000

Problem 31	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	109	109	126	0	0	0	0	0	0
N.S.	1	1.00	1.16	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.433	0.970	0.000	0.000	0.000	0.000	0.000	0.000

Problem 32	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD
size	74	74	60	0	0	0	0	0	0
N.S.	1	1.00	0.81	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.302	0.384	0.000	0.000	0.000	0.000	0.000	0.000

Problem 33	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	F	F	F	F	F	F	F(-1)
verified	N/A	Yes	N/A	N/A	TBD	TBD	TBD	TBD	TBD
size	84	84	0	0	0	0	0	0	0
N.S.	1	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.285	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Problem 34	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	F	F	F	F	F	F	F(-1)
verified	N/A	Yes	N/A	N/A	TBD	TBD	TBD	TBD	TBD
size	83	83	0	0	0	0	0	0	0
N.S.	1	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.327	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Problem 35	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	F	F	F	F	F	F	F(-1)
verified	N/A	Yes	N/A	N/A	TBD	TBD	TBD	TBD	TBD
size	84	84	0	0	0	0	0	0	0
N.S.	1	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.331	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Problem 36	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	A	A	B	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	107	114	568	112	125	217	0	205	314
N.S.	1	1.07	5.31	1.05	1.17	2.03	0.00	1.92	2.93
time (sec)	N/A	0.397	12.920	1.200	0.221	0.252	0.000	0.290	18.003

Problem 37	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	A	A	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	73	75	152	85	95	155	0	134	234
N.S.	1	1.03	2.08	1.16	1.30	2.12	0.00	1.84	3.21
time (sec)	N/A	0.246	3.441	0.717	0.228	0.246	0.000	0.286	18.594

Problem 38	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	A	A	B	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	34	34	76	42	43	77	0	74	105
N.S.	1	1.00	2.24	1.24	1.26	2.26	0.00	2.18	3.09
time (sec)	N/A	0.273	0.709	0.649	0.223	0.247	0.000	0.278	18.673

Problem 39	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	112	139	125	156	0	607	0	194	588
N.S.	1	1.24	1.12	1.39	0.00	5.42	0.00	1.73	5.25
time (sec)	N/A	1.001	2.097	0.750	0.000	0.350	0.000	0.282	19.022

Problem 40	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	84	105	144	112	0	524	0	141	515
N.S.	1	1.25	1.71	1.33	0.00	6.24	0.00	1.68	6.13
time (sec)	N/A	0.705	0.718	0.568	0.000	0.359	0.000	0.275	17.917

Problem 41	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	62	76	106	77	0	308	0	98	135
N.S.	1	1.23	1.71	1.24	0.00	4.97	0.00	1.58	2.18
time (sec)	N/A	0.488	0.367	0.467	0.000	0.300	0.000	0.288	18.108

Problem 42	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	53	62	62	53	0	245	0	63	129
N.S.	1	1.17	1.17	1.00	0.00	4.62	0.00	1.19	2.43
time (sec)	N/A	0.401	0.143	0.350	0.000	0.294	0.000	0.284	18.291

Problem 43	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	40	49	40	39	0	154	0	48	36
N.S.	1	1.22	1.00	0.98	0.00	3.85	0.00	1.20	0.90
time (sec)	N/A	0.270	0.033	0.181	0.000	0.267	0.000	0.282	18.126

Problem 44	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	57	66	59	68	0	238	0	77	184
N.S.	1	1.16	1.04	1.19	0.00	4.18	0.00	1.35	3.23
time (sec)	N/A	0.279	0.206	0.349	0.000	0.265	0.000	0.274	18.592

Problem 45	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	61	73	56	73	0	235	0	77	766
N.S.	1	1.20	0.92	1.20	0.00	3.85	0.00	1.26	12.56
time (sec)	N/A	0.358	0.272	0.463	0.000	0.278	0.000	0.274	18.997

Problem 46	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	82	103	78	112	0	285	0	112	1147
N.S.	1	1.26	0.95	1.37	0.00	3.48	0.00	1.37	13.99
time (sec)	N/A	0.667	0.362	0.543	0.000	0.265	0.000	0.268	20.241

Problem 47	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	110	136	98	145	0	329	0	149	1218
N.S.	1	1.24	0.89	1.32	0.00	2.99	0.00	1.35	11.07
time (sec)	N/A	0.915	0.791	0.721	0.000	0.264	0.000	0.272	19.017

Problem 48	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	144	178	129	234	0	410	0	252	1639
N.S.	1	1.24	0.90	1.62	0.00	2.85	0.00	1.75	11.38
time (sec)	N/A	1.271	1.452	0.981	0.000	0.284	0.000	0.276	19.318

Problem 49	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	108	142	139	168	0	493	0	158	2677
N.S.	1	1.31	1.29	1.56	0.00	4.56	0.00	1.46	24.79
time (sec)	N/A	0.633	0.650	0.550	0.000	0.275	0.000	0.268	22.691

Problem 50	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	170	225	216	314	0	933	0	297	5917
N.S.	1	1.32	1.27	1.85	0.00	5.49	0.00	1.75	34.81
time (sec)	N/A	0.993	1.971	1.050	0.000	0.300	0.000	0.286	27.173

Problem 51	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	F(-2)	B	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	239	314	279	545	0	1554	0	535	8167
N.S.	1	1.31	1.17	2.28	0.00	6.50	0.00	2.24	34.17
time (sec)	N/A	1.412	5.557	2.092	0.000	0.342	0.000	0.297	31.996

Problem 52	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	31	41	66	34	49	33	0	49	39
N.S.	1	1.32	2.13	1.10	1.58	1.06	0.00	1.58	1.26
time (sec)	N/A	0.219	0.157	0.559	0.325	0.254	0.000	0.265	19.319

Problem 53	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	68	50	67	41	71	52	0	45	27
N.S.	1	0.74	0.99	0.60	1.04	0.76	0.00	0.66	0.40
time (sec)	N/A	0.258	0.147	0.313	0.315	0.250	0.000	0.272	18.570

Problem 54	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	F	F	F	F	F	F	F(-1)
verified	N/A	Yes	N/A	N/A	TBD	TBD	TBD	TBD	TBD
size	274	274	0	0	0	0	0	0	0
N.S.	1	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.693	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Problem 55	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	F	F	F	F	F	F	F(-1)
verified	N/A	Yes	N/A	N/A	TBD	TBD	TBD	TBD	TBD
size	220	220	0	0	0	0	0	0	0
N.S.	1	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.447	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Problem 56	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	F	F	F	F	F	F	F(-1)
verified	N/A	Yes	N/A	N/A	TBD	TBD	TBD	TBD	TBD
size	104	104	0	0	0	0	0	0	0
N.S.	1	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.253	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Problem 57	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
verified	N/A	N/A	N/A	N/A	TBD	TBD	TBD	TBD	TBD
size	12	12	14	12	14	14	12	14	16
N.S.	1	1.00	1.17	1.00	1.17	1.17	1.00	1.17	1.33
time (sec)	N/A	0.165	3.158	0.613	1.236	0.256	0.495	0.324	18.725

Problem 58	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
verified	N/A	N/A	N/A	N/A	TBD	TBD	TBD	TBD	TBD
size	19	19	21	19	21	21	19	21	23
N.S.	1	1.00	1.11	1.00	1.11	1.11	1.00	1.11	1.21
time (sec)	N/A	0.194	8.296	0.677	1.655	0.259	4.478	0.396	18.431

Problem 59	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
verified	N/A	N/A	N/A	N/A	TBD	TBD	TBD	TBD	TBD
size	21	21	23	21	23	26	20	23	25
N.S.	1	1.00	1.10	1.00	1.10	1.24	0.95	1.10	1.19
time (sec)	N/A	0.201	6.528	1.039	2.105	0.255	21.256	0.396	19.495

2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi only. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio $\frac{\text{number of rules}}{\text{integrand size}}$ is also given. The larger this ratio is, the harder the integral is to solve. In this test file, problem number [51] had the largest ratio of [1.41667000000000010]

Table 2.1: Rubi specific breakdown of results for each integral

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
1	A	11	10	1.00	13	0.769
2	A	11	10	1.05	13	0.769
3	A	7	7	1.00	13	0.538
4	A	5	5	1.00	13	0.385
5	A	2	2	1.00	11	0.182
6	A	3	3	1.00	12	0.250
7	A	8	8	1.04	11	0.727
8	A	9	9	1.02	13	0.692
9	A	11	10	0.98	13	0.769
10	A	13	12	1.02	13	0.923
11	A	7	7	1.07	12	0.583
12	A	10	10	1.11	12	0.833
13	A	10	9	1.06	10	0.900
14	A	7	6	1.00	10	0.600
15	A	4	3	1.00	10	0.300
16	A	8	7	1.00	10	0.700
17	A	11	10	1.07	10	1.000
18	A	14	13	1.12	10	1.300
19	A	4	3	1.00	25	0.120
20	A	4	3	1.00	28	0.107
21	A	6	5	1.05	25	0.200
22	A	5	4	1.00	25	0.160

Continued on next page

Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
23	A	6	5	1.05	25	0.200
24	A	8	7	0.99	25	0.280
25	A	7	6	1.01	25	0.240
26	A	8	7	1.00	25	0.280
27	A	9	8	0.99	25	0.320
28	A	4	3	1.00	23	0.130
29	A	5	4	1.00	24	0.167
30	A	10	9	1.01	21	0.429
31	A	8	7	1.00	21	0.333
32	A	6	5	1.00	19	0.263
33	A	6	5	1.00	12	0.417
34	A	6	5	1.00	19	0.263
35	A	6	5	1.00	21	0.238
36	A	5	5	1.07	12	0.417
37	A	3	3	1.03	12	0.250
38	A	7	6	1.00	12	0.500
39	A	18	17	1.24	13	1.308
40	A	14	13	1.25	13	1.000
41	A	12	11	1.23	13	0.846
42	A	10	9	1.17	13	0.692
43	A	7	6	1.22	11	0.545
44	A	7	6	1.16	12	0.500
45	A	11	10	1.20	11	0.909
46	A	14	13	1.26	13	1.000
47	A	17	16	1.24	13	1.231
48	A	19	18	1.24	13	1.385
49	A	12	11	1.31	12	0.917
50	A	15	14	1.32	12	1.167
51	A	18	17	1.31	12	1.417
52	A	4	4	1.32	12	0.333
53	A	7	6	0.74	12	0.500
54	A	9	8	1.00	21	0.381
55	A	7	6	1.00	21	0.286
56	A	5	4	1.00	19	0.211

Continued on next page

Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
57	N/A	2	0	1.00	12	0.000
58	N/A	2	0	1.00	19	0.000
59	N/A	2	0	1.00	21	0.000

2.3. Detailed conclusion table specific for Rubi results

CHAPTER 3

LISTING OF INTEGRALS

3.1 $\int \frac{\csc^5(x)}{a+a\csc(x)}\,dx$

3.1.1 Optimal result

Integrand size = 13, antiderivative size = 55

$$\int \frac{\csc^5(x)}{a+a\csc(x)}\,dx = \frac{3\text{arctanh}(\cos(x))}{2a} - \frac{4\cot(x)}{a} - \frac{4\cot^3(x)}{3a} + \frac{3\cot(x)\csc(x)}{2a} + \frac{\cot(x)\csc^3(x)}{a+a\csc(x)}$$

output
```
3/2*arctanh(cos(x))/a-4*cot(x)/a-4/3*cot(x)^3/a+3/2*cot(x)*csc(x)/a+cot(x)
*csc(x)^3/(a+a*csc(x))
```

3.1.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 113 vs. 2(55) = 110.

Time = 1.10 (sec) , antiderivative size = 113, normalized size of antiderivative = 2.05

$$\int \frac{\csc^5(x)}{a+a\csc(x)}\,dx$$

$$= \frac{-20\cot\left(\frac{x}{2}\right)+3\csc^2\left(\frac{x}{2}\right)+36\log\left(\cos\left(\frac{x}{2}\right)\right)-36\log\left(\sin\left(\frac{x}{2}\right)\right)-3\sec^2\left(\frac{x}{2}\right)+8\csc^3(x)\sin^4\left(\frac{x}{2}\right)+\frac{48\sin\left(\frac{x}{2}\right.}{\cos\left(\frac{x}{2}\right)+\sin}}{24a}$$

input
```
Integrate[Csc[x]^5/(a + a*Csc[x]),x]
```

output
```
(-20*Cot[x/2] + 3*Csc[x/2]^2 + 36*Log[Cos[x/2]] - 36*Log[Sin[x/2]] - 3*Sec
[x/2]^2 + 8*Csc[x]^3*Sin[x/2]^4 + (48*Sin[x/2])/(Cos[x/2] + Sin[x/2]) - (C
sc[x/2]^4*Sin[x])/2 + 20*Tan[x/2])/(24*a)
```

3.1.3 Rubi [A] (verified)

Time = 0.45 (sec) , antiderivative size = 55, normalized size of antiderivative = 1.00, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.769, Rules used = {3042, 4305, 3042, 4274, 3042, 4254, 2009, 4255, 3042, 4257}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc^5(x)}{a\csc(x)+a}\,dx$$

$$\Big\downarrow \; 3042$$

$$\int \frac{\csc(x)^5}{a\csc(x)+a}dx$$

$$\Big\downarrow \; 4305$$

$$\frac{\cot(x)\csc^3(x)}{a\csc(x)+a}-\frac{\int \csc^3(x)(3a-4a\csc(x))dx}{a^2}$$

$$\Big\downarrow \; 3042$$

$$\frac{\cot(x)\csc^3(x)}{a\csc(x)+a}-\frac{\int \csc(x)^3(3a-4a\csc(x))dx}{a^2}$$

$$\Big\downarrow \; 4274$$

$$\frac{\cot(x)\csc^3(x)}{a\csc(x)+a}-\frac{3a\int \csc^3(x)dx-4a\int \csc^4(x)dx}{a^2}$$

$$\Big\downarrow \; 3042$$

$$\frac{\cot(x)\csc^3(x)}{a\csc(x)+a}-\frac{3a\int \csc(x)^3dx-4a\int \csc(x)^4dx}{a^2}$$

$$\Big\downarrow \; 4254$$

$$\frac{\cot(x)\csc^3(x)}{a\csc(x)+a}-\frac{4a\int \left(\cot^2(x)+1\right)d\cot(x)+3a\int \csc(x)^3dx}{a^2}$$

$$\Big\downarrow \; 2009$$

$$\frac{\cot(x)\csc^3(x)}{a\csc(x)+a} - \frac{3a\int\csc(x)^3dx + 4a\left(\frac{\cot^3(x)}{3}+\cot(x)\right)}{a^2}$$

$$\Big\downarrow \text{4255}$$

$$\frac{\cot(x)\csc^3(x)}{a\csc(x)+a} - \frac{3a\left(\frac{\int\csc(x)dx}{2} - \frac{1}{2}\cot(x)\csc(x)\right) + 4a\left(\frac{\cot^3(x)}{3}+\cot(x)\right)}{a^2}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\cot(x)\csc^3(x)}{a\csc(x)+a} - \frac{3a\left(\frac{\int\csc(x)dx}{2} - \frac{1}{2}\cot(x)\csc(x)\right) + 4a\left(\frac{\cot^3(x)}{3}+\cot(x)\right)}{a^2}$$

$$\Big\downarrow \text{4257}$$

$$\frac{\cot(x)\csc^3(x)}{a\csc(x)+a} - \frac{3a\left(-\frac{1}{2}\text{arctanh}(\cos(x)) - \frac{1}{2}\cot(x)\csc(x)\right) + 4a\left(\frac{\cot^3(x)}{3}+\cot(x)\right)}{a^2}$$

input
```
Int[Csc[x]^5/(a + a*Csc[x]),x]
```

output
```
(Cot[x]*Csc[x]^3)/(a + a*Csc[x]) - (4*a*(Cot[x] + Cot[x]^3/3) + 3*a*(-1/2*
ArcTanh[Cos[x]] - (Cot[x]*Csc[x])/2))/a^2
```

3.1.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4254
```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> Simp[-d^(-1)   Subst[Int[Exp
andIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
 d}, x] && IGtQ[n/2, 0]
```

rule 4255
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :> Simp[(-b)*Cos[c + d*
x]*((b*Csc[c + d*x])^(n - 1)/(d*(n - 1))), x] + Simp[b^2*((n - 2)/(n - 1))
  Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1]
&& IntegerQ[2*n]
```

rule 4257
```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

rule 4274
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Simp[a   Int[(d*Csc[e + f*x])^n, x], x] + Simp[b/d   In
t[(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

rule 4305
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[d^2*Cot[e + f*x]*((d*Csc[e + f*x])^(n - 2)/(f*(a +
b*Csc[e + f*x]))), x] - Simp[d^2/(a*b)   Int[(d*Csc[e + f*x])^(n - 2)*(b*(n
 - 2) - a*(n - 1)*Csc[e + f*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ
[a^2 - b^2, 0] && GtQ[n, 1]
```

3.1.4 Maple [A] (verified)

Time = 0.48 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.24

method	result	size
default	$\frac{\frac{\tan\left(\frac{x}{2}\right)^{3}}{3}-\tan\left(\frac{x}{2}\right)^{2}+7\tan\left(\frac{x}{2}\right)-\frac{1}{3\tan\left(\frac{x}{2}\right)^{3}}+\frac{1}{\tan\left(\frac{x}{2}\right)^{2}}-\frac{7}{\tan\left(\frac{x}{2}\right)}-12\ln\left(\tan\left(\frac{x}{2}\right)\right)-\frac{16}{\tan\left(\frac{x}{2}\right)+1}}{8a}$	68
parallelrisch	$-\frac{3\left(\left(\frac{\sin(4x)}{2}-\sin(2x)\right)\ln(\csc(x)-\cot(x))+\sin(3x)+\frac{5\sin(4x)}{8}-\frac{5\sin(x)}{3}-\frac{16\cos(2x)}{9}+\frac{8\cos(4x)}{9}-\frac{5\sin(2x)}{4}\right)\tan(x)}{a(-3-\cos(4x)+4\cos(2x))}$	79
risch	$-\frac{9i\,{\rm e}^{5ix}+9\,{\rm e}^{6ix}-24i\,{\rm e}^{3ix}-24\,{\rm e}^{4ix}+7i\,{\rm e}^{ix}+39\,{\rm e}^{2ix}-16}{3({\rm e}^{2ix}-1)^{3}(i+{\rm e}^{ix})a}-\frac{3\ln({\rm e}^{ix}-1)}{2a}+\frac{3\ln({\rm e}^{ix}+1)}{2a}$	99
norman	$\frac{-\frac{\tan\left(\frac{x}{2}\right)}{24a}+\frac{\tan\left(\frac{x}{2}\right)^{2}}{12a}-\frac{3\tan\left(\frac{x}{2}\right)^{3}}{4a}+\frac{3\tan\left(\frac{x}{2}\right)^{6}}{4a}-\frac{\tan\left(\frac{x}{2}\right)^{7}}{12a}+\frac{\tan\left(\frac{x}{2}\right)^{8}}{24a}-\frac{15\tan\left(\frac{x}{2}\right)^{4}}{4a}}{\tan\left(\frac{x}{2}\right)^{4}\left(\tan\left(\frac{x}{2}\right)+1\right)}-\frac{3\ln\left(\tan\left(\frac{x}{2}\right)\right)}{2a}$	103

input
```
int(csc(x)^5/(a+a*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
1/8/a*(1/3*tan(1/2*x)^3-tan(1/2*x)^2+7*tan(1/2*x)-1/3/tan(1/2*x)^3+1/tan(1
/2*x)^2-7/tan(1/2*x)-12*ln(tan(1/2*x))-16/(tan(1/2*x)+1))
```

3.1.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 168 vs. $2(49) = 98$.

Time = 0.25 (sec) , antiderivative size = 168, normalized size of antiderivative = 3.05

$$\int \frac{\csc^5(x)}{a+a\csc(x)}\,dx = \frac{32\cos(x)^4 + 14\cos(x)^3 - 48\cos(x)^2 + 9\left(\cos(x)^4 - 2\cos(x)^2 - \left(\cos(x)^3 + \cos(x)^2 - \cos(x) - 1\right)\sin\right.}{}$$

input
```
integrate(csc(x)^5/(a+a*csc(x)),x, algorithm="fricas")
```

output
```
1/12*(32*cos(x)^4 + 14*cos(x)^3 - 48*cos(x)^2 + 9*(cos(x)^4 - 2*cos(x)^2 -
 (cos(x)^3 + cos(x)^2 - cos(x) - 1)*sin(x) + 1)*log(1/2*cos(x) + 1/2) - 9*
(cos(x)^4 - 2*cos(x)^2 - (cos(x)^3 + cos(x)^2 - cos(x) - 1)*sin(x) + 1)*lo
g(-1/2*cos(x) + 1/2) + 2*(16*cos(x)^3 + 9*cos(x)^2 - 15*cos(x) - 6)*sin(x)
 - 18*cos(x) + 12)/(a*cos(x)^4 - 2*a*cos(x)^2 - (a*cos(x)^3 + a*cos(x)^2 -
 a*cos(x) - a)*sin(x) + a)
```

3.1.6 Sympy [F]

$$\int \frac{\csc^5(x)}{a+a\csc(x)}\,dx = \frac{\int \frac{\csc^5(x)}{\csc(x)+1}\,dx}{a}$$

input
```
integrate(csc(x)**5/(a+a*csc(x)),x)
```

output
```
Integral(csc(x)**5/(csc(x) + 1), x)/a
```

3.1.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 120 vs. $2(49) = 98$.

Time = 0.23 (sec) , antiderivative size = 120, normalized size of antiderivative = 2.18

$$\int \frac{\csc^5(x)}{a+a\csc(x)}\,dx = \frac{\frac{21\sin(x)}{\cos(x)+1} - \frac{3\sin(x)^2}{(\cos(x)+1)^2} + \frac{\sin(x)^3}{(\cos(x)+1)^3}}{24\,a} + \frac{\frac{2\sin(x)}{\cos(x)+1} - \frac{18\sin(x)^2}{(\cos(x)+1)^2} - \frac{69\sin(x)^3}{(\cos(x)+1)^3} - 1}{24\left(\frac{a\sin(x)^3}{(\cos(x)+1)^3} + \frac{a\sin(x)^4}{(\cos(x)+1)^4}\right)} - \frac{3\log\left(\frac{\sin(x)}{\cos(x)+1}\right)}{2\,a}$$

input
```
integrate(csc(x)^5/(a+a*csc(x)),x, algorithm="maxima")
```

output
```
1/24*(21*sin(x)/(cos(x) + 1) - 3*sin(x)^2/(cos(x) + 1)^2 + sin(x)^3/(cos(x
) + 1)^3)/a + 1/24*(2*sin(x)/(cos(x) + 1) - 18*sin(x)^2/(cos(x) + 1)^2 - 6
9*sin(x)^3/(cos(x) + 1)^3 - 1)/(a*sin(x)^3/(cos(x) + 1)^3 + a*sin(x)^4/(co
s(x) + 1)^4) - 3/2*log(sin(x)/(cos(x) + 1))/a
```

3.1.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 96, normalized size of antiderivative = 1.75

$$\int \frac{\csc^5(x)}{a + a\csc(x)}\,dx = -\frac{3\log\left(\left|\tan\left(\frac{1}{2}x\right)\right|\right)}{2a} + \frac{a^2\tan\left(\frac{1}{2}x\right)^3 - 3a^2\tan\left(\frac{1}{2}x\right)^2 + 21a^2\tan\left(\frac{1}{2}x\right)}{24a^3} - \frac{2}{a\left(\tan\left(\frac{1}{2}x\right) + 1\right)} + \frac{66\tan\left(\frac{1}{2}x\right)^3 - 21\tan\left(\frac{1}{2}x\right)^2 + 3\tan\left(\frac{1}{2}x\right) - 1}{24a\tan\left(\frac{1}{2}x\right)^3}$$

input
```
integrate(csc(x)^5/(a+a*csc(x)),x, algorithm="giac")
```

output
```
-3/2*log(abs(tan(1/2*x)))/a + 1/24*(a^2*tan(1/2*x)^3 - 3*a^2*tan(1/2*x)^2
+ 21*a^2*tan(1/2*x))/a^3 - 2/(a*(tan(1/2*x) + 1)) + 1/24*(66*tan(1/2*x)^3
- 21*tan(1/2*x)^2 + 3*tan(1/2*x) - 1)/(a*tan(1/2*x)^3)
```

3.1.9 Mupad [B] (verification not implemented)

Time = 17.69 (sec) , antiderivative size = 89, normalized size of antiderivative = 1.62

$$\int \frac{\csc^5(x)}{a + a\csc(x)}\,dx = \frac{7\tan\left(\frac{x}{2}\right)}{8a} - \frac{23\tan\left(\frac{x}{2}\right)^3 + 6\tan\left(\frac{x}{2}\right)^2 - \frac{2\tan\left(\frac{x}{2}\right)}{3} + \frac{1}{3}}{8a\tan\left(\frac{x}{2}\right)^4 + 8a\tan\left(\frac{x}{2}\right)^3} - \frac{\tan\left(\frac{x}{2}\right)^2}{8a} + \frac{\tan\left(\frac{x}{2}\right)^3}{24a} - \frac{3\ln\left(\tan\left(\frac{x}{2}\right)\right)}{2a}$$

input
```
int(1/(sin(x)^5*(a + a/sin(x))),x)
```

output
```
(7*tan(x/2))/(8*a) - (6*tan(x/2)^2 - (2*tan(x/2))/3 + 23*tan(x/2)^3 + 1/3)
/(8*a*tan(x/2)^3 + 8*a*tan(x/2)^4) - tan(x/2)^2/(8*a) + tan(x/2)^3/(24*a)
- (3*log(tan(x/2)))/(2*a)
```

3.2 $\int \frac{\csc^4(x)}{a+a\csc(x)}\,dx$

3.2.1 Optimal result

Integrand size = 13, antiderivative size = 44

$$\int \frac{\csc^4(x)}{a+a\csc(x)}\,dx=-\frac{3\text{arctanh}(\cos(x))}{2a}+\frac{2\cot(x)}{a}-\frac{3\cot(x)\csc(x)}{2a}+\frac{\cot(x)\csc^2(x)}{a+a\csc(x)}$$

output
```
-3/2*arctanh(cos(x))/a+2*cot(x)/a-3/2*cot(x)*csc(x)/a+cot(x)*csc(x)^2/(a+a
*csc(x))
```

3.2.2 Mathematica [A] (verified)

Time = 0.48 (sec) , antiderivative size = 83, normalized size of antiderivative = 1.89

$$\int \frac{\csc^4(x)}{a+a\csc(x)}\,dx$$
$$=\frac{4\cot\left(\frac{x}{2}\right)-\csc^2\left(\frac{x}{2}\right)-12\log\left(\cos\left(\frac{x}{2}\right)\right)+12\log\left(\sin\left(\frac{x}{2}\right)\right)+\sec^2\left(\frac{x}{2}\right)-\frac{16\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}-4\tan\left(\frac{x}{2}\right)}{8a}$$

input
```
Integrate[Csc[x]^4/(a + a*Csc[x]),x]
```

output
```
(4*Cot[x/2] - Csc[x/2]^2 - 12*Log[Cos[x/2]] + 12*Log[Sin[x/2]] + Sec[x/2]^
2 - (16*Sin[x/2])/(Cos[x/2] + Sin[x/2]) - 4*Tan[x/2])/(8*a)
```

3.2.3 Rubi [A] (verified)

Time = 0.43 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.05, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.769, Rules used = {3042, 4305, 3042, 4274, 3042, 4254, 24, 4255, 3042, 4257}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc^4(x)}{a\csc(x)+a}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\csc(x)^4}{a\csc(x)+a}dx$$

$$\Big\downarrow \text{4305}$$

$$\frac{\cot(x)\csc^2(x)}{a\csc(x)+a}-\frac{\int \csc^2(x)(2a-3a\csc(x))dx}{a^2}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\cot(x)\csc^2(x)}{a\csc(x)+a}-\frac{\int \csc(x)^2(2a-3a\csc(x))dx}{a^2}$$

$$\Big\downarrow \text{4274}$$

$$\frac{\cot(x)\csc^2(x)}{a\csc(x)+a}-\frac{2a\int \csc^2(x)dx-3a\int \csc^3(x)dx}{a^2}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\cot(x)\csc^2(x)}{a\csc(x)+a}-\frac{2a\int \csc(x)^2dx-3a\int \csc(x)^3dx}{a^2}$$

$$\Big\downarrow \text{4254}$$

$$\frac{\cot(x)\csc^2(x)}{a\csc(x)+a}-\frac{-2a\int 1d\cot(x)-3a\int \csc(x)^3dx}{a^2}$$

$$\Big\downarrow \text{24}$$

$$\frac{\cot(x)\csc^2(x)}{a\csc(x)+a}-\frac{-3a\int \csc(x)^3dx-2a\cot(x)}{a^2}$$

$$\Big\downarrow \text{4255}$$

$$\frac{\cot(x)\csc^2(x)}{a\csc(x)+a}-\frac{-3a\left(\frac{\int \csc(x)dx}{2}-\frac{1}{2}\cot(x)\csc(x)\right)-2a\cot(x)}{a^2}$$

$$\Big\downarrow \ 3042$$

$$\frac{\cot(x)\csc^2(x)}{a\csc(x)+a}-\frac{-3a\left(\frac{\int \csc(x)dx}{2}-\frac{1}{2}\cot(x)\csc(x)\right)-2a\cot(x)}{a^2}$$

$$\Big\downarrow \ 4257$$

$$\frac{\cot(x)\csc^2(x)}{a\csc(x)+a}-\frac{-3a\left(-\frac{1}{2}\text{arctanh}(\cos(x))-\frac{1}{2}\cot(x)\csc(x)\right)-2a\cot(x)}{a^2}$$

input
```
Int[Csc[x]^4/(a + a*Csc[x]),x]
```

output
```
(Cot[x]*Csc[x]^2)/(a + a*Csc[x]) - (-2*a*Cot[x] - 3*a*(-1/2*ArcTanh[Cos[x]
] - (Cot[x]*Csc[x])/2))/a^2
```

3.2.3.1 Defintions of rubi rules used

rule 24
```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4254
```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> Simp[-d^(-1)   Subst[Int[Exp
andIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
 d}, x] && IGtQ[n/2, 0]
```

rule 4255
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :> Simp[(-b)*Cos[c + d*
x]*((b*Csc[c + d*x])^(n - 1)/(d*(n - 1))), x] + Simp[b^2*((n - 2)/(n - 1))
  Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1]
&& IntegerQ[2*n]
```

rule 4257
```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

rule 4274
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Simp[a   Int[(d*Csc[e + f*x])^n, x], x] + Simp[b/d   In
t[(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

rule 4305
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[d^2*Cot[e + f*x]*((d*Csc[e + f*x])^(n - 2)/(f*(a +
b*Csc[e + f*x]))), x] - Simp[d^2/(a*b)   Int[(d*Csc[e + f*x])^(n - 2)*(b*(n
 - 2) - a*(n - 1)*Csc[e + f*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ
[a^2 - b^2, 0] && GtQ[n, 1]
```

3.2.4 Maple [A] (verified)

Time = 0.41 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.23

method	result	size
default	$\frac{\frac{\tan\left(\frac{x}{2}\right)^{2}}{2}-2\tan\left(\frac{x}{2}\right)+\frac{8}{\tan\left(\frac{x}{2}\right)+1}-\frac{1}{2\tan\left(\frac{x}{2}\right)^{2}}+\frac{2}{\tan\left(\frac{x}{2}\right)}+6\ln\left(\tan\left(\frac{x}{2}\right)\right)}{4a}$	54
parallelrisch	$\frac{(3\cos(2x)-3)\ln(\csc(x)-\cot(x))+6\cos(x)-4\sec(x)+4\tan(x)+3\cos(2x)-4\sin(2x)-3}{2a(-1+\cos(2x))}$	57
norman	$\frac{\frac{3\tan\left(\frac{x}{2}\right)^{3}}{a}-\frac{\tan\left(\frac{x}{2}\right)}{8a}+\frac{3\tan\left(\frac{x}{2}\right)^{2}}{8a}-\frac{3\tan\left(\frac{x}{2}\right)^{5}}{8a}+\frac{\tan\left(\frac{x}{2}\right)^{6}}{8a}}{\tan\left(\frac{x}{2}\right)^{3}\left(\tan\left(\frac{x}{2}\right)+1\right)}+\frac{3\ln\left(\tan\left(\frac{x}{2}\right)\right)}{2a}$	81
risch	$\frac{-5\,\mathrm{e}^{2ix}+3i\mathrm{e}^{3ix}+3\,\mathrm{e}^{4ix}+4-i\mathrm{e}^{ix}}{\left(\mathrm{e}^{2ix}-1\right)^{2}\left(i+\mathrm{e}^{ix}\right)a}+\frac{3\ln\left(\mathrm{e}^{ix}-1\right)}{2a}-\frac{3\ln\left(\mathrm{e}^{ix}+1\right)}{2a}$	83

input
```
int(csc(x)^4/(a+a*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
1/4/a*(1/2*tan(1/2*x)^2-2*tan(1/2*x)+8/(tan(1/2*x)+1)-1/2/tan(1/2*x)^2+2/t
an(1/2*x)+6*ln(tan(1/2*x)))
```

3.2.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 134 vs. 2(40) = 80.

Time = 0.24 (sec) , antiderivative size = 134, normalized size of antiderivative = 3.05

$$\int \frac{\csc^4(x)}{a+a\csc(x)}\,dx$$
$$= \frac{8\,\cos(x)^3+6\,\cos(x)^2-3\left(\cos(x)^3+\cos(x)^2+\left(\cos(x)^2-1\right)\sin(x)-\cos(x)-1\right)\log\left(\frac{1}{2}\,\cos(x)+\frac{1}{2}\right)}{4\left(a\cos(x)^3+a\,\mathrm{c}\right.}$$

input
```
integrate(csc(x)^4/(a+a*csc(x)),x, algorithm="fricas")
```

output
```
1/4*(8*cos(x)^3 + 6*cos(x)^2 - 3*(cos(x)^3 + cos(x)^2 + (cos(x)^2 - 1)*sin
(x) - cos(x) - 1)*log(1/2*cos(x) + 1/2) + 3*(cos(x)^3 + cos(x)^2 + (cos(x)
^2 - 1)*sin(x) - cos(x) - 1)*log(-1/2*cos(x) + 1/2) - 2*(4*cos(x)^2 + cos(
x) - 2)*sin(x) - 6*cos(x) - 4)/(a*cos(x)^3 + a*cos(x)^2 - a*cos(x) + (a*co
s(x)^2 - a)*sin(x) - a)
```

3.2.6 Sympy [F]

$$\int \frac{\csc^4(x)}{a + a\csc(x)}\,dx = \frac{\int \frac{\csc^4(x)}{\csc(x)+1}\,dx}{a}$$

input
```
integrate(csc(x)**4/(a+a*csc(x)),x)
```

output
```
Integral(csc(x)**4/(csc(x) + 1), x)/a
```

3.2.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 97 vs. $2(40) = 80$.

Time = 0.22 (sec) , antiderivative size = 97, normalized size of antiderivative = 2.20

$$\int \frac{\csc^4(x)}{a + a\csc(x)}\,dx = -\frac{\frac{4\sin(x)}{\cos(x)+1} - \frac{\sin(x)^2}{(\cos(x)+1)^2}}{8\,a} + \frac{\frac{3\sin(x)}{\cos(x)+1} + \frac{20\sin(x)^2}{(\cos(x)+1)^2} - 1}{8\left(\frac{a\sin(x)^2}{(\cos(x)+1)^2} + \frac{a\sin(x)^3}{(\cos(x)+1)^3}\right)} + \frac{3\log\left(\frac{\sin(x)}{\cos(x)+1}\right)}{2\,a}$$

input
```
integrate(csc(x)^4/(a+a*csc(x)),x, algorithm="maxima")
```

output
```
-1/8*(4*sin(x)/(cos(x) + 1) - sin(x)^2/(cos(x) + 1)^2)/a + 1/8*(3*sin(x)/(
cos(x) + 1) + 20*sin(x)^2/(cos(x) + 1)^2 - 1)/(a*sin(x)^2/(cos(x) + 1)^2 +
 a*sin(x)^3/(cos(x) + 1)^3) + 3/2*log(sin(x)/(cos(x) + 1))/a
```

3.2.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 73, normalized size of antiderivative = 1.66

$$\int \frac{\csc^4(x)}{a + a\csc(x)}\,dx = \frac{3\,\log\left(\left|\tan\left(\frac{1}{2}\,x\right)\right|\right)}{2\,a} + \frac{a\tan\left(\frac{1}{2}\,x\right)^2 - 4\,a\tan\left(\frac{1}{2}\,x\right)}{8\,a^2} + \frac{2}{a\left(\tan\left(\frac{1}{2}\,x\right) + 1\right)} - \frac{18\,\tan\left(\frac{1}{2}\,x\right)^2 - 4\,\tan\left(\frac{1}{2}\,x\right) + 1}{8\,a\tan\left(\frac{1}{2}\,x\right)^2}$$

input
```
integrate(csc(x)^4/(a+a*csc(x)),x, algorithm="giac")
```

output
```
3/2*log(abs(tan(1/2*x)))/a + 1/8*(a*tan(1/2*x)^2 - 4*a*tan(1/2*x))/a^2 + 2
/(a*(tan(1/2*x) + 1)) - 1/8*(18*tan(1/2*x)^2 - 4*tan(1/2*x) + 1)/(a*tan(1/
2*x)^2)
```

3.2.9 Mupad [B] (verification not implemented)

Time = 19.44 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.57

$$\int \frac{\csc^4(x)}{a + a\csc(x)}\,dx = \frac{10\tan\left(\frac{x}{2}\right)^2 + \frac{3\tan\left(\frac{x}{2}\right)}{2} - \frac{1}{2}}{4\,a\tan\left(\frac{x}{2}\right)^3 + 4\,a\tan\left(\frac{x}{2}\right)^2} - \frac{\tan\left(\frac{x}{2}\right)}{2\,a} + \frac{\tan\left(\frac{x}{2}\right)^2}{8\,a} + \frac{3\,\ln\left(\tan\left(\frac{x}{2}\right)\right)}{2\,a}$$

input
```
int(1/(sin(x)^4*(a + a/sin(x))),x)
```

output
```
((3*tan(x/2))/2 + 10*tan(x/2)^2 - 1/2)/(4*a*tan(x/2)^2 + 4*a*tan(x/2)^3) -
 tan(x/2)/(2*a) + tan(x/2)^2/(8*a) + (3*log(tan(x/2)))/(2*a)
```

3.3 $\int \frac{\csc^3(x)}{a+a\csc(x)}\,dx$

3.3.1 Optimal result

Integrand size = 13, antiderivative size = 27

$$\int \frac{\csc^3(x)}{a+a\csc(x)}\,dx=\frac{\operatorname{arctanh}(\cos(x))}{a}-\frac{\cot(x)}{a}-\frac{\cot(x)}{a+a\csc(x)}$$

output
```
arctanh(cos(x))/a-cot(x)/a-cot(x)/(a+a*csc(x))
```

3.3.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 63 vs. 2(27) = 54.

Time = 0.29 (sec) , antiderivative size = 63, normalized size of antiderivative = 2.33

$$\int \frac{\csc^3(x)}{a+a\csc(x)}\,dx=\frac{-\cot\left(\frac{x}{2}\right)+2\log\left(\cos\left(\frac{x}{2}\right)\right)-2\log\left(\sin\left(\frac{x}{2}\right)\right)+\frac{4\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}+\tan\left(\frac{x}{2}\right)}{2a}$$

input
```
Integrate[Csc[x]^3/(a + a*Csc[x]),x]
```

output
```
(-Cot[x/2] + 2*Log[Cos[x/2]] - 2*Log[Sin[x/2]] + (4*Sin[x/2])/(Cos[x/2] +
Sin[x/2]) + Tan[x/2])/(2*a)
```

3.3.3 Rubi [A] (verified)

Time = 0.38 (sec) , antiderivative size = 27, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.538, Rules used = {3042, 4277, 3042, 4276, 3042, 4257, 4281}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc^3(x)}{a\csc(x)+a}dx$$
$$\Big\downarrow \text{3042}$$
$$\int \frac{\csc(x)^3}{a\csc(x)+a}dx$$
$$\Big\downarrow \text{4277}$$
$$-\int \frac{\csc^2(x)}{\csc(x)a+a}dx-\frac{\cot(x)}{a}$$
$$\Big\downarrow \text{3042}$$
$$-\int \frac{\csc(x)^2}{\csc(x)a+a}dx-\frac{\cot(x)}{a}$$
$$\Big\downarrow \text{4276}$$
$$-\frac{\int \csc(x)dx}{a}+\int \frac{\csc(x)}{\csc(x)a+a}dx-\frac{\cot(x)}{a}$$
$$\Big\downarrow \text{3042}$$
$$-\frac{\int \csc(x)dx}{a}+\int \frac{\csc(x)}{\csc(x)a+a}dx-\frac{\cot(x)}{a}$$
$$\Big\downarrow \text{4257}$$
$$\int \frac{\csc(x)}{\csc(x)a+a}dx+\frac{\operatorname{arctanh}(\cos(x))}{a}-\frac{\cot(x)}{a}$$
$$\Big\downarrow \text{4281}$$
$$\frac{\operatorname{arctanh}(\cos(x))}{a}-\frac{\cot(x)}{a}-\frac{\cot(x)}{a\csc(x)+a}$$

input
```
Int[Csc[x]^3/(a + a*Csc[x]),x]
```

output
```
ArcTanh[Cos[x]]/a - Cot[x]/a - Cot[x]/(a + a*Csc[x])
```

3.3.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4257
```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

rule 4276
```
Int[csc[(e_.) + (f_.)*(x_)]^2/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> Simp[1/b   Int[Csc[e + f*x], x], x] - Simp[a/b   Int[Csc[e + f*x]/(
a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

rule 4277
```
Int[csc[(e_.) + (f_.)*(x_)]^3/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> Simp[-Cot[e + f*x]/(b*f), x] - Simp[a/b   Int[Csc[e + f*x]^2/(a + b
*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

rule 4281
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[-Cot[e + f*x]/(f*(b + a*Csc[e + f*x])), x] /; FreeQ[{a, b, e, f}
, x] && EqQ[a^2 - b^2, 0]
```

3.3.4 Maple [A] (verified)

Time = 0.35 (sec) , antiderivative size = 36, normalized size of antiderivative = 1.33

method	result	size
default	$\frac{\tan\left(\frac{x}{2}\right)-\frac{4}{\tan\left(\frac{x}{2}\right)+1}-\frac{1}{\tan\left(\frac{x}{2}\right)}-2\ln\left(\tan\left(\frac{x}{2}\right)\right)}{2a}$	36
parallelrisch	$\frac{(-2\cos(2x)+2)\ln(\csc(x)-\cot(x))+(4\tan(x)-3)\cos(2x)+4\sin(x)\tan(x)+3}{2a(-1+\cos(2x))}$	50
norman	$\frac{-\frac{3\tan\left(\frac{x}{2}\right)^{2}}{a}-\frac{\tan\left(\frac{x}{2}\right)}{2a}+\frac{\tan\left(\frac{x}{2}\right)^{4}}{2a}}{\tan\left(\frac{x}{2}\right)^{2}\left(\tan\left(\frac{x}{2}\right)+1\right)}-\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{a}$	59
risch	$-\frac{2\left(e^{2ix}-2+ie^{ix}\right)}{\left(e^{2ix}-1\right)\left(i+e^{ix}\right)a}-\frac{\ln\left(e^{ix}-1\right)}{a}+\frac{\ln\left(e^{ix}+1\right)}{a}$	66

input
```
int(csc(x)^3/(a+a*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
1/2/a*(tan(1/2*x)-4/(tan(1/2*x)+1)-1/tan(1/2*x)-2*ln(tan(1/2*x)))
```

3.3.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 91 vs. $2(27) = 54$.

Time = 0.24 (sec) , antiderivative size = 91, normalized size of antiderivative = 3.37

$$\int \frac{\csc^3(x)}{a + a\csc(x)}\,dx$$
$$= \frac{4\cos(x)^2 + \left(\cos(x)^2 - (\cos(x)+1)\sin(x) - 1\right)\log\left(\frac{1}{2}\cos(x) + \frac{1}{2}\right) - \left(\cos(x)^2 - (\cos(x)+1)\sin(x) - \right.}{2\left(a\cos(x)^2 - (a\cos(x)+a)\sin(x) - a\right.}$$

input
```
integrate(csc(x)^3/(a+a*csc(x)),x, algorithm="fricas")
```

output
```
1/2*(4*cos(x)^2 + (cos(x)^2 - (cos(x) + 1)*sin(x) - 1)*log(1/2*cos(x) + 1/
2) - (cos(x)^2 - (cos(x) + 1)*sin(x) - 1)*log(-1/2*cos(x) + 1/2) + 2*(2*co
s(x) + 1)*sin(x) + 2*cos(x) - 2)/(a*cos(x)^2 - (a*cos(x) + a)*sin(x) - a)
```

3.3.6 Sympy [F]

$$\int \frac{\csc^3(x)}{a + a\csc(x)}\,dx = \frac{\int \frac{\csc^3(x)}{\csc(x)+1}\,dx}{a}$$

input
```
integrate(csc(x)**3/(a+a*csc(x)),x)
```

output
```
Integral(csc(x)**3/(csc(x) + 1), x)/a
```

3.3.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 68 vs. 2(27) = 54.

Time = 0.22 (sec) , antiderivative size = 68, normalized size of antiderivative = 2.52

$$\int \frac{\csc^3(x)}{a + a\csc(x)}\,dx = -\frac{\frac{5\,\sin(x)}{\cos(x)+1} + 1}{2\left(\frac{a\sin(x)}{\cos(x)+1} + \frac{a\sin(x)^2}{(\cos(x)+1)^2}\right)} - \frac{\log\left(\frac{\sin(x)}{\cos(x)+1}\right)}{a} + \frac{\sin(x)}{2\,a(\cos(x)+1)}$$

input
```
integrate(csc(x)^3/(a+a*csc(x)),x, algorithm="maxima")
```

output
```
-1/2*(5*sin(x)/(cos(x) + 1) + 1)/(a*sin(x)/(cos(x) + 1) + a*sin(x)^2/(cos(
x) + 1)^2) - log(sin(x)/(cos(x) + 1))/a + 1/2*sin(x)/(a*(cos(x) + 1))
```

3.3.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 53, normalized size of antiderivative = 1.96

$$\int \frac{\csc^3(x)}{a + a\csc(x)}\,dx = -\frac{\log\left(\left|\tan\left(\frac{1}{2}\,x\right)\right|\right)}{a} + \frac{\tan\left(\frac{1}{2}\,x\right)}{2\,a} + \frac{\tan\left(\frac{1}{2}\,x\right)^2 - 4\,\tan\left(\frac{1}{2}\,x\right) - 1}{2\left(\tan\left(\frac{1}{2}\,x\right)^2 + \tan\left(\frac{1}{2}\,x\right)\right)a}$$

input
```
integrate(csc(x)^3/(a+a*csc(x)),x, algorithm="giac")
```

output
```
-log(abs(tan(1/2*x)))/a + 1/2*tan(1/2*x)/a + 1/2*(tan(1/2*x)^2 - 4*tan(1/2
*x) - 1)/((tan(1/2*x)^2 + tan(1/2*x))*a)
```

3.3.9 Mupad [B] (verification not implemented)

Time = 18.41 (sec) , antiderivative size = 49, normalized size of antiderivative = 1.81

$$\int \frac{\csc^3(x)}{a + a\csc(x)}\,dx = \frac{\tan\left(\frac{x}{2}\right)}{2\,a} - \frac{5\tan\left(\frac{x}{2}\right) + 1}{2\,a\tan\left(\frac{x}{2}\right)^2 + 2\,a\tan\left(\frac{x}{2}\right)} - \frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{a}$$

input
```
int(1/(sin(x)^3*(a + a/sin(x))),x)
```

output
```
tan(x/2)/(2*a) - (5*tan(x/2) + 1)/(2*a*tan(x/2) + 2*a*tan(x/2)^2) - log(ta
n(x/2))/a
```

3.4 $\int \frac{\csc^2(x)}{a+a\csc(x)}\,dx$

3.4.1 Optimal result

Integrand size = 13, antiderivative size = 20

$$\int \frac{\csc^2(x)}{a+a\csc(x)}\,dx=-\frac{\operatorname{arctanh}(\cos(x))}{a}+\frac{\cot(x)}{a+a\csc(x)}$$

output
```
-arctanh(cos(x))/a+cot(x)/(a+a*csc(x))
```

3.4.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 44 vs. 2(20) = 40.

Time = 0.14 (sec) , antiderivative size = 44, normalized size of antiderivative = 2.20

$$\int \frac{\csc^2(x)}{a+a\csc(x)}\,dx=\frac{-\log\left(\cos\left(\frac{x}{2}\right)\right)+\log\left(\sin\left(\frac{x}{2}\right)\right)-\frac{2\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}}{a}$$

input
```
Integrate[Csc[x]^2/(a + a*Csc[x]),x]
```

output
```
(-Log[Cos[x/2]] + Log[Sin[x/2]] - (2*Sin[x/2])/(Cos[x/2] + Sin[x/2]))/a
```

3.4.3 Rubi [A] (verified)

Time = 0.29 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.385, Rules used = {3042, 4276, 3042, 4257, 4281}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc^2(x)}{a\csc(x)+a}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\csc(x)^2}{a\csc(x)+a}dx$$

$$\Big\downarrow \text{4276}$$

$$\frac{\int \csc(x)dx}{a}-\int \frac{\csc(x)}{\csc(x)a+a}dx$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \csc(x)dx}{a}-\int \frac{\csc(x)}{\csc(x)a+a}dx$$

$$\Big\downarrow \text{4257}$$

$$-\int \frac{\csc(x)}{\csc(x)a+a}dx-\frac{\text{arctanh}(\cos(x))}{a}$$

$$\Big\downarrow \text{4281}$$

$$\frac{\cot(x)}{a\csc(x)+a}-\frac{\text{arctanh}(\cos(x))}{a}$$

input
```
Int[Csc[x]^2/(a + a*Csc[x]),x]
```

output
```
-(ArcTanh[Cos[x]]/a) + Cot[x]/(a + a*Csc[x])
```

3.4.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4257
```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

rule 4276
```
Int[csc[(e_.) + (f_.)*(x_)]^2/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> Simp[1/b   Int[Csc[e + f*x], x], x] - Simp[a/b   Int[Csc[e + f*x]/(
a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

rule 4281
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[-Cot[e + f*x]/(f*(b + a*Csc[e + f*x])), x] /; FreeQ[{a, b, e, f}
, x] && EqQ[a^2 - b^2, 0]
```

3.4.4 Maple [A] (verified)

Time = 0.20 (sec) , antiderivative size = 21, normalized size of antiderivative = 1.05

method	result	size
default	$\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)+\frac{2}{\tan\left(\frac{x}{2}\right)+1}}{a}$	21
norman	$\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}+\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{a}$	24
parallelrisch	$\frac{2+\ln\left(\tan\left(\frac{x}{2}\right)\right)\left(\tan\left(\frac{x}{2}\right)+1\right)}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}$	27
risch	$\frac{2}{\left(i+e^{ix}\right)a}-\frac{\ln\left(e^{ix}+1\right)}{a}+\frac{\ln\left(e^{ix}-1\right)}{a}$	42

input
```
int(csc(x)^2/(a+a*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
1/a*(ln(tan(1/2*x))+2/(tan(1/2*x)+1))
```

3.4.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 53 vs. 2(20) = 40.

Time = 0.25 (sec) , antiderivative size = 53, normalized size of antiderivative = 2.65

$$\int \frac{\csc^2(x)}{a+a\csc(x)}\,dx = -\frac{(\cos(x)+\sin(x)+1)\log\left(\frac{1}{2}\cos(x)+\frac{1}{2}\right)-(\cos(x)+\sin(x)+1)\log\left(-\frac{1}{2}\cos(x)+\frac{1}{2}\right)-2\cos(x)+2}{2\left(a\cos(x)+a\sin(x)+a\right)}$$

input
```
integrate(csc(x)^2/(a+a*csc(x)),x, algorithm="fricas")
```

output
```
-1/2*((cos(x) + sin(x) + 1)*log(1/2*cos(x) + 1/2) - (cos(x) + sin(x) + 1)*
log(-1/2*cos(x) + 1/2) - 2*cos(x) + 2*sin(x) - 2)/(a*cos(x) + a*sin(x) + a
)
```

3.4.6 Sympy [F]

$$\int \frac{\csc^2(x)}{a+a\csc(x)}\,dx = \frac{\int \frac{\csc^2(x)}{\csc(x)+1}\,dx}{a}$$

input
```
integrate(csc(x)**2/(a+a*csc(x)),x)
```

output
```
Integral(csc(x)**2/(csc(x) + 1), x)/a
```

3.4.7 Maxima [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 31, normalized size of antiderivative = 1.55

$$\int \frac{\csc^2(x)}{a+a\csc(x)}\,dx = \frac{\log\left(\frac{\sin(x)}{\cos(x)+1}\right)}{a} + \frac{2}{a+\frac{a\sin(x)}{\cos(x)+1}}$$

input
```
integrate(csc(x)^2/(a+a*csc(x)),x, algorithm="maxima")
```

output
```
log(sin(x)/(cos(x) + 1))/a + 2/(a + a*sin(x)/(cos(x) + 1))
```

3.4.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 24, normalized size of antiderivative = 1.20

$$\int \frac{\csc^2(x)}{a + a\csc(x)}\,dx = \frac{\log\left(\left|\tan\left(\frac{1}{2}\,x\right)\right|\right)}{a} + \frac{2}{a\left(\tan\left(\frac{1}{2}\,x\right) + 1\right)}$$

input
```
integrate(csc(x)^2/(a+a*csc(x)),x, algorithm="giac")
```

output
```
log(abs(tan(1/2*x)))/a + 2/(a*(tan(1/2*x) + 1))
```

3.4.9 Mupad [B] (verification not implemented)

Time = 18.04 (sec) , antiderivative size = 23, normalized size of antiderivative = 1.15

$$\int \frac{\csc^2(x)}{a + a\csc(x)}\,dx = \frac{2}{a\,\left(\tan\left(\frac{x}{2}\right) + 1\right)} + \frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{a}$$

input
```
int(1/(sin(x)^2*(a + a/sin(x))),x)
```

output
```
2/(a*(tan(x/2) + 1)) + log(tan(x/2))/a
```

3.5 $\int \frac{\csc(x)}{a+a\csc(x)}\,dx$

3.5.1 Optimal result

Integrand size = 11, antiderivative size = 12

$$\int \frac{\csc(x)}{a+a\csc(x)}\,dx=-\frac{\cot(x)}{a+a\csc(x)}$$

output
```
-cot(x)/(a+a*csc(x))
```

3.5.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 26 vs. 2(12) = 24.

Time = 0.09 (sec) , antiderivative size = 26, normalized size of antiderivative = 2.17

$$\int \frac{\csc(x)}{a+a\csc(x)}\,dx=\frac{2\sin\left(\frac{x}{2}\right)}{a\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)}$$

input
```
Integrate[Csc[x]/(a + a*Csc[x]),x]
```

output
```
(2*Sin[x/2])/(a*(Cos[x/2] + Sin[x/2]))
```

3.5.3 Rubi [A] (verified)

Time = 0.18 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.182$, Rules used = {3042, 4281}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(x)}{a\csc(x)+a}\,dx$$

$$\Big\downarrow \text{ 3042}$$

$$\int \frac{\csc(x)}{a\csc(x)+a}dx$$

$$\Big\downarrow \text{ 4281}$$

$$-\frac{\cot(x)}{a\csc(x)+a}$$

input
```
Int[Csc[x]/(a + a*Csc[x]),x]
```

output
```
-(Cot[x]/(a + a*Csc[x]))
```

3.5.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4281
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[-Cot[e + f*x]/(f*(b + a*Csc[e + f*x])), x] /; FreeQ[{a, b, e, f}
, x] && EqQ[a^2 - b^2, 0]
```

3.5.4 Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.17

method	result	size
default	$-\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}$	14
norman	$-\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}$	14
parallelrisch	$-\frac{2}{a\left(\tan\left(\frac{x}{2}\right)+1\right)}$	14
risch	$-\frac{2}{\left(i+e^{ix}\right)a}$	16

input
```
int(csc(x)/(a+a*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
-2/a/(tan(1/2*x)+1)
```

3.5.5 Fricas [A] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 22, normalized size of antiderivative = 1.83

$$\int \frac{\csc(x)}{a+a\csc(x)}\,dx = -\frac{\cos(x) - \sin(x) + 1}{a\cos(x) + a\sin(x) + a}$$

input
```
integrate(csc(x)/(a+a*csc(x)),x, algorithm="fricas")
```

output
```
-(cos(x) - sin(x) + 1)/(a*cos(x) + a*sin(x) + a)
```

3.5.6 Sympy [F]

$$\int \frac{\csc(x)}{a+a\csc(x)}\,dx = \frac{\int \frac{\csc(x)}{\csc(x)+1}\,dx}{a}$$

input
```
integrate(csc(x)/(a+a*csc(x)),x)
```

output
```
Integral(csc(x)/(csc(x) + 1), x)/a
```

3.5.7 Maxima [A] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 16, normalized size of antiderivative = 1.33

$$\int \frac{\csc(x)}{a + a\csc(x)}\, dx = -\frac{2}{a + \frac{a\sin(x)}{\cos(x)+1}}$$

input
```
integrate(csc(x)/(a+a*csc(x)),x, algorithm="maxima")
```

output
```
-2/(a + a*sin(x)/(cos(x) + 1))
```

3.5.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 13, normalized size of antiderivative = 1.08

$$\int \frac{\csc(x)}{a + a\csc(x)}\, dx = -\frac{2}{a\left(\tan\left(\frac{1}{2}x\right) + 1\right)}$$

input
```
integrate(csc(x)/(a+a*csc(x)),x, algorithm="giac")
```

output
```
-2/(a*(tan(1/2*x) + 1))
```

3.5.9 Mupad [B] (verification not implemented)

Time = 17.55 (sec) , antiderivative size = 13, normalized size of antiderivative = 1.08

$$\int \frac{\csc(x)}{a + a\csc(x)}\, dx = -\frac{2}{a\,\left(\tan\left(\frac{x}{2}\right) + 1\right)}$$

input
```
int(1/(sin(x)*(a + a/sin(x))),x)
```

output
```
-2/(a*(tan(x/2) + 1))
```

3.6 $\int \frac{1}{a+a\csc(c+dx)}\,dx$

3.6.1 Optimal result

Integrand size = 12, antiderivative size = 28

$$\int \frac{1}{a+a\csc(c+dx)}\,dx=\frac{x}{a}+\frac{\cot(c+dx)}{d(a+a\csc(c+dx))}$$

output
```
x/a+cot(d*x+c)/d/(a+a*csc(d*x+c))
```

3.6.2 Mathematica [A] (verified)

Time = 0.26 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.68

$$\int \frac{1}{a+a\csc(c+dx)}\,dx=\frac{c+dx-\frac{2\sin\left(\frac{1}{2}(c+dx)\right)}{\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)}}{ad}$$

input
```
Integrate[(a + a*Csc[c + d*x])^(-1),x]
```

output
```
(c + d*x - (2*Sin[(c + d*x)/2])/(Cos[(c + d*x)/2] + Sin[(c + d*x)/2]))/(a*
d)
```

3.6.3 Rubi [A] (verified)

Time = 0.18 (sec) , antiderivative size = 28, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {3042, 4264, 24}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{a\csc(c+dx)+a}\,dx$$

$$\Big\downarrow \ 3042$$

$$\int \frac{1}{a\csc(c+dx)+a}dx$$

$$\Big\downarrow \ 4264$$

$$\frac{\cot(c+dx)}{d(a\csc(c+dx)+a)}-\frac{\int -adx}{a^2}$$

$$\Big\downarrow \ 24$$

$$\frac{\cot(c+dx)}{d(a\csc(c+dx)+a)}+\frac{x}{a}$$

input
```
Int[(a + a*Csc[c + d*x])^(-1),x]
```

output
```
x/a + Cot[c + d*x]/(d*(a + a*Csc[c + d*x]))
```

3.6.3.1 Defintions of rubi rules used

rule 24
```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4264
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(-Cot[c
+ d*x])*((a + b*Csc[c + d*x])^n/(d*(2*n + 1))), x] + Simp[1/(a^2*(2*n + 1))
   Int[(a + b*Csc[c + d*x])^(n + 1)*(a*(2*n + 1) - b*(n + 1)*Csc[c + d*x]),
 x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && LeQ[n, -1] && Int
egerQ[2*n]
```

3.6.4 Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.28 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.04

method	result	size
risch	$\frac{x}{a}+\frac{2}{da\left(i+{\mathrm e}^{i(dx+c)}\right)}$	29
derivativedivides	$\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)+\frac{4}{2\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+2}}{da}$	37
default	$\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)+\frac{4}{2\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+2}}{da}$	37
parallelrisch	$\frac{\tan\left(\frac{dx}{2}+\frac{c}{2}\right)xd+dx-2\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{da\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)}$	48
norman	$\frac{\frac{x}{a}+\frac{x\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{a}-\frac{2\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{da}}{\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1}$	52

input
```
int(1/(a+a*csc(d*x+c)),x,method=_RETURNVERBOSE)
```

output
```
x/a+2/d/a/(I+exp(I*(d*x+c)))
```

3.6.5 Fricas [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.93

$$\int \frac{1}{a+a\csc(c+dx)}\,dx=\frac{dx+(dx+1)\cos(dx+c)+(dx-1)\sin(dx+c)+1}{ad\cos(dx+c)+ad\sin(dx+c)+ad}$$

input
```
integrate(1/(a+a*csc(d*x+c)),x, algorithm="fricas")
```

output
```
(d*x + (d*x + 1)*cos(d*x + c) + (d*x - 1)*sin(d*x + c) + 1)/(a*d*cos(d*x +
 c) + a*d*sin(d*x + c) + a*d)
```

3.6.6 Sympy [F]

$$\int \frac{1}{a+a\csc(c+dx)}\,dx=\frac{\int \frac{1}{\csc(c+dx)+1}\,dx}{a}$$

input
```
integrate(1/(a+a*csc(d*x+c)),x)
```

output
```
Integral(1/(csc(c + d*x) + 1), x)/a
```

3.6.7 Maxima [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 50, normalized size of antiderivative = 1.79

$$\int \frac{1}{a+a\csc(c+dx)}\,dx=\frac{2\left(\frac{\arctan\left(\frac{\sin(dx+c)}{\cos(dx+c)+1}\right)}{a}+\frac{1}{a+\frac{a\sin(dx+c)}{\cos(dx+c)+1}}\right)}{d}$$

input
```
integrate(1/(a+a*csc(d*x+c)),x, algorithm="maxima")
```

output
```
2*(arctan(sin(d*x + c)/(cos(d*x + c) + 1))/a + 1/(a + a*sin(d*x + c)/(cos(
d*x + c) + 1)))/d
```

3.6.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 32, normalized size of antiderivative = 1.14

$$\int \frac{1}{a+a\csc(c+dx)}\,dx=\frac{\frac{dx+c}{a}+\frac{2}{a\left(\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+1\right)}}{d}$$

input
```
integrate(1/(a+a*csc(d*x+c)),x, algorithm="giac")
```

output
```
((d*x + c)/a + 2/(a*(tan(1/2*d*x + 1/2*c) + 1)))/d
```

3.6.9 Mupad [B] (verification not implemented)

Time = 18.40 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.96

$$\int \frac{1}{a + a\csc(c + dx)}\,dx = \frac{x}{a} + \frac{2}{a\,d\,\left(\tan\left(\frac{c}{2} + \frac{d\,x}{2}\right) + 1\right)}$$

input
```
int(1/(a + a/sin(c + d*x)),x)
```

output
```
x/a + 2/(a*d*(tan(c/2 + (d*x)/2) + 1))
```

3.7 $\int \frac{\sin(x)}{a+a\csc(x)}\,dx$

3.7.1 Optimal result

Integrand size = 11, antiderivative size = 25

$$\int \frac{\sin(x)}{a+a\csc(x)}\,dx=-\frac{x}{a}-\frac{2\cos(x)}{a}+\frac{\cos(x)}{a+a\csc(x)}$$

output
```
-x/a-2*cos(x)/a+cos(x)/(a+a*csc(x))
```

3.7.2 Mathematica [A] (verified)

Time = 0.19 (sec) , antiderivative size = 32, normalized size of antiderivative = 1.28

$$\int \frac{\sin(x)}{a+a\csc(x)}\,dx=-\frac{x+\cos(x)-\frac{2\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}}{a}$$

input
```
Integrate[Sin[x]/(a + a*Csc[x]),x]
```

output
```
-((x + Cos[x] - (2*Sin[x/2])/(Cos[x/2] + Sin[x/2]))/a)
```

3.7.3 Rubi [A] (verified)

Time = 0.30 (sec) , antiderivative size = 26, normalized size of antiderivative = 1.04, number of steps used = 8, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.727, Rules used = {3042, 4306, 25, 3042, 4274, 24, 3042, 3118}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sin(x)}{a\csc(x)+a}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\csc(x)(a\csc(x)+a)}dx$$

$$\Big\downarrow \text{4306}$$

$$\frac{\cos(x)}{a\csc(x)+a}-\frac{\int -((2a-a\csc(x))\sin(x))dx}{a^2}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int (2a-a\csc(x))\sin(x)dx}{a^2}+\frac{\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{2a-a\csc(x)}{\csc(x)}dx}{a^2}+\frac{\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{4274}$$

$$\frac{2a\int \sin(x)dx-a\int 1dx}{a^2}+\frac{\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{24}$$

$$\frac{2a\int \sin(x)dx-ax}{a^2}+\frac{\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{2a\int \sin(x)dx-ax}{a^2}+\frac{\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3118}$$

$$\frac{-ax-2a\cos(x)}{a^2}+\frac{\cos(x)}{a\csc(x)+a}$$

input
```
Int[Sin[x]/(a + a*Csc[x]),x]
```

output
```
(-(a*x) - 2*a*Cos[x])/a^2 + Cos[x]/(a + a*Csc[x])
```

3.7.3.1 Defintions of rubi rules used

rule 24
```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3118
```
Int[sin[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-Cos[c + d*x]/d, x] /; FreeQ
[{c, d}, x]
```

rule 4274
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Simp[a   Int[(d*Csc[e + f*x])^n, x], x] + Simp[b/d   In
t[(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

rule 4306
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[Cot[e + f*x]*((d*Csc[e + f*x])^n/(f*(a + b*Csc[e +
f*x]))), x] - Simp[1/a^2   Int[(d*Csc[e + f*x])^n*(a*(n - 1) - b*n*Csc[e +
f*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && LtQ[n, 0
]
```

3.7.4 Maple [A] (verified)

Time = 0.34 (sec) , antiderivative size = 20, normalized size of antiderivative = 0.80

method	result	size
parallelrisch	$\frac{-\cos(x)-x-2+\tan(x)-\sec(x)}{a}$	20
default	$\frac{-\frac{2}{\tan(\frac{x}{2})+1}-\frac{2}{1+\tan(\frac{x}{2})^2}-2\arctan(\tan(\frac{x}{2}))}{a}$	36
risch	$-\frac{x}{a}-\frac{e^{ix}}{2a}-\frac{e^{-ix}}{2a}-\frac{2}{(i+e^{ix})a}$	43
norman	$\frac{-\frac{4}{a}-\frac{2\tan(\frac{x}{2})}{a}-\frac{2\tan(\frac{x}{2})^2}{a}-\frac{x}{a}-\frac{x\tan(\frac{x}{2})}{a}-\frac{x\tan(\frac{x}{2})^2}{a}-\frac{x\tan(\frac{x}{2})^3}{a}}{(1+\tan(\frac{x}{2})^2)(\tan(\frac{x}{2})+1)}$	86

input
```
int(sin(x)/(a+a*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
(-cos(x)-x-2+tan(x)-sec(x))/a
```

3.7.5 Fricas [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 35, normalized size of antiderivative = 1.40

$$\int \frac{\sin(x)}{a+a\csc(x)}\,dx = -\frac{(x+2)\cos(x)+\cos(x)^2+(x+\cos(x)-1)\sin(x)+x+1}{a\cos(x)+a\sin(x)+a}$$

input
```
integrate(sin(x)/(a+a*csc(x)),x, algorithm="fricas")
```

output
```
-((x + 2)*cos(x) + cos(x)^2 + (x + cos(x) - 1)*sin(x) + x + 1)/(a*cos(x) +
 a*sin(x) + a)
```

3.7.6 Sympy [F]

$$\int \frac{\sin(x)}{a+a\csc(x)}\,dx = \frac{\int \frac{\sin(x)}{\csc(x)+1}\,dx}{a}$$

input
```
integrate(sin(x)/(a+a*csc(x)),x)
```

output
```
Integral(sin(x)/(csc(x) + 1), x)/a
```

3.7.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 78 vs. 2(25) = 50.

Time = 0.32 (sec) , antiderivative size = 78, normalized size of antiderivative = 3.12

$$\int \frac{\sin(x)}{a + a\csc(x)}\, dx = -\frac{2\left(\frac{\sin(x)}{\cos(x)+1} + \frac{\sin(x)^2}{(\cos(x)+1)^2} + 2\right)}{a + \frac{a\sin(x)}{\cos(x)+1} + \frac{a\sin(x)^2}{(\cos(x)+1)^2} + \frac{a\sin(x)^3}{(\cos(x)+1)^3}} - \frac{2\arctan\left(\frac{\sin(x)}{\cos(x)+1}\right)}{a}$$

input
```
integrate(sin(x)/(a+a*csc(x)),x, algorithm="maxima")
```

output
```
-2*(sin(x)/(cos(x) + 1) + sin(x)^2/(cos(x) + 1)^2 + 2)/(a + a*sin(x)/(cos(
x) + 1) + a*sin(x)^2/(cos(x) + 1)^2 + a*sin(x)^3/(cos(x) + 1)^3) - 2*arcta
n(sin(x)/(cos(x) + 1))/a
```

3.7.8 Giac [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 44, normalized size of antiderivative = 1.76

$$\int \frac{\sin(x)}{a + a\csc(x)}\, dx = -\frac{x}{a} - \frac{2\left(\tan\left(\frac{1}{2}x\right)^2 + \tan\left(\frac{1}{2}x\right) + 2\right)}{\left(\tan\left(\frac{1}{2}x\right)^3 + \tan\left(\frac{1}{2}x\right)^2 + \tan\left(\frac{1}{2}x\right) + 1\right)a}$$

input
```
integrate(sin(x)/(a+a*csc(x)),x, algorithm="giac")
```

output
```
-x/a - 2*(tan(1/2*x)^2 + tan(1/2*x) + 2)/((tan(1/2*x)^3 + tan(1/2*x)^2 + t
an(1/2*x) + 1)*a)
```

3.7.9 Mupad [B] (verification not implemented)

Time = 18.34 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.84

$$\int \frac{\sin(x)}{a + a\csc(x)}\, dx = -\frac{x}{a} - \frac{2\tan\left(\frac{x}{2}\right)^2 + 2\tan\left(\frac{x}{2}\right) + 4}{a\left(\tan\left(\frac{x}{2}\right)^2 + 1\right)\left(\tan\left(\frac{x}{2}\right) + 1\right)}$$

input
```
int(sin(x)/(a + a/sin(x)),x)
```

output
```
- x/a - (2*tan(x/2) + 2*tan(x/2)^2 + 4)/(a*(tan(x/2)^2 + 1)*(tan(x/2) + 1)
)
```

3.7. $\int \frac{\sin(x)}{a+a\csc(x)}\,dx$

3.8 $\int \frac{\sin^2(x)}{a+a\csc(x)}\,dx$

3.8.1 Optimal result

Integrand size = 13, antiderivative size = 40

$$\int \frac{\sin^2(x)}{a+a\csc(x)}\,dx=\frac{3x}{2a}+\frac{2\cos(x)}{a}-\frac{3\cos(x)\sin(x)}{2a}+\frac{\cos(x)\sin(x)}{a+a\csc(x)}$$

output
```
3/2*x/a+2*cos(x)/a-3/2*cos(x)*sin(x)/a+cos(x)*sin(x)/(a+a*csc(x))
```

3.8.2 Mathematica [A] (verified)

Time = 0.25 (sec) , antiderivative size = 42, normalized size of antiderivative = 1.05

$$\int \frac{\sin^2(x)}{a+a\csc(x)}\,dx=-\frac{-6x-4\cos(x)+\frac{8\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}+\sin(2x)}{4a}$$

input
```
Integrate[Sin[x]^2/(a + a*Csc[x]),x]
```

output
```
-1/4*(-6*x - 4*Cos[x] + (8*Sin[x/2])/(Cos[x/2] + Sin[x/2]) + Sin[2*x])/a
```

3.8.3 Rubi [A] (verified)

Time = 0.35 (sec) , antiderivative size = 41, normalized size of antiderivative = 1.02, number of steps used = 9, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.692, Rules used = {3042, 4306, 25, 3042, 4274, 3042, 3115, 24, 3118}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sin^2(x)}{a\csc(x)+a}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\csc(x)^2(a\csc(x)+a)}dx$$

$$\Big\downarrow \text{4306}$$

$$\frac{\sin(x)\cos(x)}{a\csc(x)+a}-\frac{\int -\left((3a-2a\csc(x))\sin^2(x)\right)dx}{a^2}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int (3a-2a\csc(x))\sin^2(x)dx}{a^2}+\frac{\sin(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{3a-2a\csc(x)}{\csc(x)^2}dx}{a^2}+\frac{\sin(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{4274}$$

$$\frac{3a\int \sin^2(x)dx-2a\int \sin(x)dx}{a^2}+\frac{\sin(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{3a\int \sin(x)^2dx-2a\int \sin(x)dx}{a^2}+\frac{\sin(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3115}$$

$$\frac{3a\left(\frac{\int 1dx}{2}-\frac{1}{2}\sin(x)\cos(x)\right)-2a\int \sin(x)dx}{a^2}+\frac{\sin(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{24}$$

$$\frac{3a\left(\frac{x}{2}-\frac{1}{2}\sin(x)\cos(x)\right)-2a\int \sin(x)dx}{a^2}+\frac{\sin(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3118}$$

$$\frac{2a\cos(x)+3a\left(\frac{x}{2}-\frac{1}{2}\sin(x)\cos(x)\right)}{a^2}+\frac{\sin(x)\cos(x)}{a\csc(x)+a}$$

input
```
Int[Sin[x]^2/(a + a*Csc[x]),x]
```

output
```
(Cos[x]*Sin[x])/(a + a*Csc[x]) + (2*a*Cos[x] + 3*a*(x/2 - (Cos[x]*Sin[x])/
2))/a^2
```

3.8.3.1 Defintions of rubi rules used

rule 24
```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3115
```
Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :> Simp[(-b)*Cos[c + d*
x]*((b*Sin[c + d*x])^(n - 1)/(d*n)), x] + Simp[b^2*((n - 1)/n)   Int[(b*Sin
[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[
2*n]
```

rule 3118
```
Int[sin[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-Cos[c + d*x]/d, x] /; FreeQ
[{c, d}, x]
```

rule 4274
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Simp[a   Int[(d*Csc[e + f*x])^n, x], x] + Simp[b/d   In
t[(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

rule 4306
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[Cot[e + f*x]*((d*Csc[e + f*x])^n/(f*(a + b*Csc[e +
f*x]))), x] - Simp[1/a^2   Int[(d*Csc[e + f*x])^n*(a*(n - 1) - b*n*Csc[e +
f*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && LtQ[n, 0
]
```

3.8.4 Maple [A] (verified)

Time = 0.40 (sec) , antiderivative size = 29, normalized size of antiderivative = 0.72

method	result	size
parallelrisch	$\frac{6x+8-\sin(2x)+4\cos(x)+4\sec(x)-4\tan(x)}{4a}$	29
risch	$\frac{3x}{2a}+\frac{e^{ix}}{2a}+\frac{e^{-ix}}{2a}+\frac{2}{(i+e^{ix})a}-\frac{\sin(2x)}{4a}$	52
default	$\frac{\frac{2\left(\frac{\tan\left(\frac{x}{2}\right)^3}{2}+\tan\left(\frac{x}{2}\right)^2-\frac{\tan\left(\frac{x}{2}\right)}{2}+1\right)}{\left(1+\tan\left(\frac{x}{2}\right)^2\right)^2}+3\arctan\left(\tan\left(\frac{x}{2}\right)\right)+\frac{16}{8\tan\left(\frac{x}{2}\right)+8}}{a}$	58
norman	$\frac{\frac{3}{a}-\frac{\tan\left(\frac{x}{2}\right)^5}{a}+\frac{2\tan\left(\frac{x}{2}\right)^4}{a}+\frac{\tan\left(\frac{x}{2}\right)^3}{a}+\frac{3\tan\left(\frac{x}{2}\right)^2}{a}+\frac{3x}{2a}+\frac{3x\tan\left(\frac{x}{2}\right)}{2a}+\frac{3x\tan\left(\frac{x}{2}\right)^2}{a}+\frac{3x\tan\left(\frac{x}{2}\right)^3}{a}+\frac{3x\tan\left(\frac{x}{2}\right)^4}{2a}+\frac{3x\tan\left(\frac{x}{2}\right)^5}{2a}}{\left(1+\tan\left(\frac{x}{2}\right)^2\right)^2\left(\tan\left(\frac{x}{2}\right)+1\right)}$	133

input
```
int(sin(x)^2/(a+a*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
1/4*(6*x+8-sin(2*x)+4*cos(x)+4*sec(x)-4*tan(x))/a
```

3.8.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 53, normalized size of antiderivative = 1.32

$$\int \frac{\sin^2(x)}{a+a\csc(x)}\,dx$$
$$=\frac{\cos(x)^3+3(x+1)\cos(x)+2\cos(x)^2-\left(\cos(x)^2-3x-\cos(x)+2\right)\sin(x)+3x+2}{2(a\cos(x)+a\sin(x)+a)}$$

input
```
integrate(sin(x)^2/(a+a*csc(x)),x, algorithm="fricas")
```

output
```
1/2*(cos(x)^3 + 3*(x + 1)*cos(x) + 2*cos(x)^2 - (cos(x)^2 - 3*x - cos(x) +
 2)*sin(x) + 3*x + 2)/(a*cos(x) + a*sin(x) + a)
```

3.8.6 Sympy [F]

$$\int \frac{\sin^2(x)}{a + a\csc(x)}\,dx = \frac{\int \frac{\sin^2(x)}{\csc(x)+1}\,dx}{a}$$

input
```
integrate(sin(x)**2/(a+a*csc(x)),x)
```

output
```
Integral(sin(x)**2/(csc(x) + 1), x)/a
```

3.8.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 128 vs. 2(36) = 72.

Time = 0.34 (sec) , antiderivative size = 128, normalized size of antiderivative = 3.20

$$\int \frac{\sin^2(x)}{a + a\csc(x)}\,dx = \frac{\frac{\sin(x)}{\cos(x)+1} + \frac{5\sin(x)^2}{(\cos(x)+1)^2} + \frac{3\sin(x)^3}{(\cos(x)+1)^3} + \frac{3\sin(x)^4}{(\cos(x)+1)^4} + 4}{a + \frac{a\sin(x)}{\cos(x)+1} + \frac{2a\sin(x)^2}{(\cos(x)+1)^2} + \frac{2a\sin(x)^3}{(\cos(x)+1)^3} + \frac{a\sin(x)^4}{(\cos(x)+1)^4} + \frac{a\sin(x)^5}{(\cos(x)+1)^5}} + \frac{3\arctan\left(\frac{\sin(x)}{\cos(x)+1}\right)}{a}$$

input
```
integrate(sin(x)^2/(a+a*csc(x)),x, algorithm="maxima")
```

output
```
(sin(x)/(cos(x) + 1) + 5*sin(x)^2/(cos(x) + 1)^2 + 3*sin(x)^3/(cos(x) + 1)
^3 + 3*sin(x)^4/(cos(x) + 1)^4 + 4)/(a + a*sin(x)/(cos(x) + 1) + 2*a*sin(x
)^2/(cos(x) + 1)^2 + 2*a*sin(x)^3/(cos(x) + 1)^3 + a*sin(x)^4/(cos(x) + 1)
^4 + a*sin(x)^5/(cos(x) + 1)^5) + 3*arctan(sin(x)/(cos(x) + 1))/a
```

3.8.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 56, normalized size of antiderivative = 1.40

$$\int \frac{\sin^2(x)}{a + a\csc(x)}\,dx = \frac{3x}{2a} + \frac{\tan\left(\frac{1}{2}x\right)^3 + 2\tan\left(\frac{1}{2}x\right)^2 - \tan\left(\frac{1}{2}x\right) + 2}{\left(\tan\left(\frac{1}{2}x\right)^2 + 1\right)^2 a} + \frac{2}{a\left(\tan\left(\frac{1}{2}x\right) + 1\right)}$$

input
```
integrate(sin(x)^2/(a+a*csc(x)),x, algorithm="giac")
```

output
```
3/2*x/a + (tan(1/2*x)^3 + 2*tan(1/2*x)^2 - tan(1/2*x) + 2)/((tan(1/2*x)^2
+ 1)^2*a) + 2/(a*(tan(1/2*x) + 1))
```

3.8.9 Mupad [B] (verification not implemented)

Time = 18.30 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.48

$$\int \frac{\sin^2(x)}{a+a\csc(x)}\,dx = \frac{3x}{2a} + \frac{3\tan\left(\frac{x}{2}\right)^4 + 3\tan\left(\frac{x}{2}\right)^3 + 5\tan\left(\frac{x}{2}\right)^2 + \tan\left(\frac{x}{2}\right) + 4}{a\left(\tan\left(\frac{x}{2}\right)^2 + 1\right)^2\left(\tan\left(\frac{x}{2}\right) + 1\right)}$$

input
```
int(sin(x)^2/(a + a/sin(x)),x)
```

output
```
(3*x)/(2*a) + (tan(x/2) + 5*tan(x/2)^2 + 3*tan(x/2)^3 + 3*tan(x/2)^4 + 4)/
(a*(tan(x/2)^2 + 1)^2*(tan(x/2) + 1))
```

3.9 $\int \frac{\sin^3(x)}{a+a\csc(x)}\,dx$

3.9.1 Optimal result

Integrand size = 13, antiderivative size = 53

$$\int \frac{\sin^3(x)}{a+a\csc(x)}\,dx=-\frac{3x}{2a}-\frac{4\cos(x)}{a}+\frac{4\cos^3(x)}{3a}+\frac{3\cos(x)\sin(x)}{2a}+\frac{\cos(x)\sin^2(x)}{a+a\csc(x)}$$

output
```
-3/2*x/a-4*cos(x)/a+4/3*cos(x)^3/a+3/2*cos(x)*sin(x)/a+cos(x)*sin(x)^2/(a+
a*csc(x))
```

3.9.2 Mathematica [A] (verified)

Time = 0.32 (sec) , antiderivative size = 49, normalized size of antiderivative = 0.92

$$\int \frac{\sin^3(x)}{a+a\csc(x)}\,dx=\frac{-21\cos(x)+\cos(3x)+3\left(-6x+\frac{8\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}+\sin(2x)\right)}{12a}$$

input
```
Integrate[Sin[x]^3/(a + a*Csc[x]),x]
```

output
```
(-21*Cos[x] + Cos[3*x] + 3*(-6*x + (8*Sin[x/2])/(Cos[x/2] + Sin[x/2]) + Si
n[2*x]))/(12*a)
```

3.9.3 Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.98, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.769, Rules used = {3042, 4306, 25, 3042, 4274, 3042, 3113, 2009, 3115, 24}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sin^3(x)}{a\csc(x)+a}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\csc(x)^3(a\csc(x)+a)}dx$$

$$\Big\downarrow \text{4306}$$

$$\frac{\sin^2(x)\cos(x)}{a\csc(x)+a}-\frac{\int -\left((4a-3a\csc(x))\sin^3(x)\right)dx}{a^2}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int (4a-3a\csc(x))\sin^3(x)dx}{a^2}+\frac{\sin^2(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{4a-3a\csc(x)}{\csc(x)^3}dx}{a^2}+\frac{\sin^2(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{4274}$$

$$\frac{4a\int \sin^3(x)dx-3a\int \sin^2(x)dx}{a^2}+\frac{\sin^2(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{4a\int \sin(x)^3dx-3a\int \sin(x)^2dx}{a^2}+\frac{\sin^2(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3113}$$

$$\frac{-3a\int \sin(x)^2dx-4a\int \left(1-\cos^2(x)\right)d\cos(x)}{a^2}+\frac{\sin^2(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{2009}$$

$$\frac{-3a\int \sin(x)^2dx-4a\left(\cos(x)-\frac{\cos^3(x)}{3}\right)}{a^2}+\frac{\sin^2(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3115}$$

$$\frac{-3a\left(\frac{\int 1dx}{2}-\frac{1}{2}\sin(x)\cos(x)\right)-4a\left(\cos(x)-\frac{\cos^3(x)}{3}\right)}{a^2}+\frac{\sin^2(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{24}$$

$$\frac{-4a\left(\cos(x)-\frac{\cos^3(x)}{3}\right)-3a\left(\frac{x}{2}-\frac{1}{2}\sin(x)\cos(x)\right)}{a^2}+\frac{\sin^2(x)\cos(x)}{a\csc(x)+a}$$

input
```
Int[Sin[x]^3/(a + a*Csc[x]),x]
```

output
```
(Cos[x]*Sin[x]^2)/(a + a*Csc[x]) + (-4*a*(Cos[x] - Cos[x]^3/3) - 3*a*(x/2
- (Cos[x]*Sin[x])/2))/a^2
```

3.9.3.1 Defintions of rubi rules used

rule 24
```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3113
```
Int[sin[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> Simp[-d^(-1)   Subst[Int[Exp
and[(1 - x^2)^((n - 1)/2), x], x], x, Cos[c + d*x]], x] /; FreeQ[{c, d}, x]
 && IGtQ[(n - 1)/2, 0]
```

rule 3115
```
Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :> Simp[(-b)*Cos[c + d*
x]*((b*Sin[c + d*x])^(n - 1)/(d*n)), x] + Simp[b^2*((n - 1)/n)   Int[(b*Sin
[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[
2*n]
```

rule 4274
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Simp[a   Int[(d*Csc[e + f*x])^n, x], x] + Simp[b/d   In
t[(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

rule 4306
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[Cot[e + f*x]*((d*Csc[e + f*x])^n/(f*(a + b*Csc[e +
f*x]))), x] - Simp[1/a^2   Int[(d*Csc[e + f*x])^n*(a*(n - 1) - b*n*Csc[e +
f*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && LtQ[n, 0
]
```

3.9.4 Maple [A] (verified)

Time = 0.46 (sec) , antiderivative size = 33, normalized size of antiderivative = 0.62

method	result
parallelrisch	$\frac{-18x-32+\cos(3x)-21\cos(x)+3\sin(2x)+12\tan(x)-12\sec(x)}{12a}$
risch	$-\frac{3x}{2a}-\frac{7\,\mathrm{e}^{ix}}{8a}-\frac{7\,\mathrm{e}^{-ix}}{8a}-\frac{2}{(i+\mathrm{e}^{ix})a}+\frac{\cos(3x)}{12a}+\frac{\sin(2x)}{4a}$
default	$\frac{-\frac{2}{\tan(\frac{x}{2})+1}-\frac{2\left(\frac{\tan(\frac{x}{2})^5}{2}+\tan(\frac{x}{2})^4+4\tan(\frac{x}{2})^2-\frac{\tan(\frac{x}{2})}{2}+\frac{5}{3}\right)}{\left(1+\tan(\frac{x}{2})^2\right)^3}-3\arctan(\tan(\frac{x}{2}))}{a}$
norman	$\frac{-\frac{5\tan(\frac{x}{2})^2}{a}+\frac{5\tan(\frac{x}{2})^5}{a}-\frac{3x}{2a}-\frac{8}{3a}-\frac{3x\tan(\frac{x}{2})}{2a}-\frac{9x\tan(\frac{x}{2})^2}{2a}-\frac{9x\tan(\frac{x}{2})^3}{2a}-\frac{9x\tan(\frac{x}{2})^4}{2a}-\frac{9x\tan(\frac{x}{2})^5}{2a}-\frac{3x\tan(\frac{x}{2})^6}{2a}-\frac{3x\tan(\frac{x}{2})^7}{2a}-}{\left(1+\tan(\frac{x}{2})^2\right)^3(\tan(\frac{x}{2})+1)}$

input
```
int(sin(x)^3/(a+a*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
1/12*(-18*x-32+cos(3*x)-21*cos(x)+3*sin(2*x)+12*tan(x)-12*sec(x))/a
```

3.9.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 70, normalized size of antiderivative = 1.32

$$\int \frac{\sin^3(x)}{a+a\csc(x)}\,dx$$
$$=\frac{2\,\cos(x)^4-\cos(x)^3-3\,(3\,x+5)\cos(x)-12\,\cos(x)^2+\left(2\,\cos(x)^3+3\,\cos(x)^2-9\,x-9\,\cos(x)+6\right)\mathrm{s}}{6\,(a\cos(x)+a\sin(x)+a)}$$

input
```
integrate(sin(x)^3/(a+a*csc(x)),x, algorithm="fricas")
```

output
```
1/6*(2*cos(x)^4 - cos(x)^3 - 3*(3*x + 5)*cos(x) - 12*cos(x)^2 + (2*cos(x)^
3 + 3*cos(x)^2 - 9*x - 9*cos(x) + 6)*sin(x) - 9*x - 6)/(a*cos(x) + a*sin(x
) + a)
```

3.9.6 Sympy [F]

$$\int \frac{\sin^3(x)}{a + a\csc(x)}\,dx = \frac{\int \frac{\sin^3(x)}{\csc(x)+1}\,dx}{a}$$

input
```
integrate(sin(x)**3/(a+a*csc(x)),x)
```

output
```
Integral(sin(x)**3/(csc(x) + 1), x)/a
```

3.9.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 180 vs. $2(47) = 94$.

Time = 0.35 (sec) , antiderivative size = 180, normalized size of antiderivative = 3.40

$$\int \frac{\sin^3(x)}{a + a\csc(x)}\,dx = -\frac{\frac{7\sin(x)}{\cos(x)+1} + \frac{39\sin(x)^2}{(\cos(x)+1)^2} + \frac{24\sin(x)^3}{(\cos(x)+1)^3} + \frac{24\sin(x)^4}{(\cos(x)+1)^4} + \frac{9\sin(x)^5}{(\cos(x)+1)^5} + \frac{9\sin(x)^6}{(\cos(x)+1)^6} + 16}{3\left(a + \frac{a\sin(x)}{\cos(x)+1} + \frac{3a\sin(x)^2}{(\cos(x)+1)^2} + \frac{3a\sin(x)^3}{(\cos(x)+1)^3} + \frac{3a\sin(x)^4}{(\cos(x)+1)^4} + \frac{3a\sin(x)^5}{(\cos(x)+1)^5} + \frac{a\sin(x)^6}{(\cos(x)+1)^6} + \frac{a\sin(x)^7}{(\cos(x)+1)^7}\right)} - \frac{3\arctan\left(\frac{\sin(x)}{\cos(x)+1}\right)}{a}$$

input
```
integrate(sin(x)^3/(a+a*csc(x)),x, algorithm="maxima")
```

output
```
-1/3*(7*sin(x)/(cos(x) + 1) + 39*sin(x)^2/(cos(x) + 1)^2 + 24*sin(x)^3/(co
s(x) + 1)^3 + 24*sin(x)^4/(cos(x) + 1)^4 + 9*sin(x)^5/(cos(x) + 1)^5 + 9*s
in(x)^6/(cos(x) + 1)^6 + 16)/(a + a*sin(x)/(cos(x) + 1) + 3*a*sin(x)^2/(co
s(x) + 1)^2 + 3*a*sin(x)^3/(cos(x) + 1)^3 + 3*a*sin(x)^4/(cos(x) + 1)^4 +
3*a*sin(x)^5/(cos(x) + 1)^5 + a*sin(x)^6/(cos(x) + 1)^6 + a*sin(x)^7/(cos(
x) + 1)^7) - 3*arctan(sin(x)/(cos(x) + 1))/a
```

3.9.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 67, normalized size of antiderivative = 1.26

$$\int \frac{\sin^3(x)}{a+a\csc(x)}\,dx = -\frac{3x}{2a} - \frac{2}{a\left(\tan\left(\frac{1}{2}x\right)+1\right)} - \frac{3\tan\left(\frac{1}{2}x\right)^5 + 6\tan\left(\frac{1}{2}x\right)^4 + 24\tan\left(\frac{1}{2}x\right)^2 - 3\tan\left(\frac{1}{2}x\right) + 10}{3\left(\tan\left(\frac{1}{2}x\right)^2+1\right)^3 a}$$

input
```
integrate(sin(x)^3/(a+a*csc(x)),x, algorithm="giac")
```

output
```
-3/2*x/a - 2/(a*(tan(1/2*x) + 1)) - 1/3*(3*tan(1/2*x)^5 + 6*tan(1/2*x)^4 +
 24*tan(1/2*x)^2 - 3*tan(1/2*x) + 10)/((tan(1/2*x)^2 + 1)^3*a)
```

3.9.9 Mupad [B] (verification not implemented)

Time = 19.65 (sec) , antiderivative size = 78, normalized size of antiderivative = 1.47

$$\int \frac{\sin^3(x)}{a+a\csc(x)}\,dx = -\frac{3x}{2a} - \frac{3\tan\left(\frac{x}{2}\right)^6 + 3\tan\left(\frac{x}{2}\right)^5 + 8\tan\left(\frac{x}{2}\right)^4 + 8\tan\left(\frac{x}{2}\right)^3 + 13\tan\left(\frac{x}{2}\right)^2 + \frac{7\tan\left(\frac{x}{2}\right)}{3} + \frac{16}{3}}{a\left(\tan\left(\frac{x}{2}\right)^2+1\right)^3\left(\tan\left(\frac{x}{2}\right)+1\right)}$$

input
```
int(sin(x)^3/(a + a/sin(x)),x)
```

output
```
- (3*x)/(2*a) - ((7*tan(x/2))/3 + 13*tan(x/2)^2 + 8*tan(x/2)^3 + 8*tan(x/2
)^4 + 3*tan(x/2)^5 + 3*tan(x/2)^6 + 16/3)/(a*(tan(x/2)^2 + 1)^3*(tan(x/2)
+ 1))
```

3.10 $\int \frac{\sin^4(x)}{a+a\csc(x)}\,dx$

3.10.1 Optimal result

Integrand size = 13, antiderivative size = 66

$$\int \frac{\sin^4(x)}{a+a\csc(x)}\,dx=\frac{15x}{8a}+\frac{4\cos(x)}{a}-\frac{4\cos^3(x)}{3a}-\frac{15\cos(x)\sin(x)}{8a}-\frac{5\cos(x)\sin^3(x)}{4a}+\frac{\cos(x)\sin^3(x)}{a+a\csc(x)}$$

output
```
15/8*x/a+4*cos(x)/a-4/3*cos(x)^3/a-15/8*cos(x)*sin(x)/a-5/4*cos(x)*sin(x)^
3/a+cos(x)*sin(x)^3/(a+a*csc(x))
```

3.10.2 Mathematica [A] (verified)

Time = 0.38 (sec) , antiderivative size = 57, normalized size of antiderivative = 0.86

$$\int \frac{\sin^4(x)}{a+a\csc(x)}\,dx=\frac{168\cos(x)-8\cos(3x)+3\left(60x-\frac{64\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}-16\sin(2x)+\sin(4x)\right)}{96a}$$

input
```
Integrate[Sin[x]^4/(a + a*Csc[x]),x]
```

output
```
(168*Cos[x] - 8*Cos[3*x] + 3*(60*x - (64*Sin[x/2])/(Cos[x/2] + Sin[x/2]) -
 16*Sin[2*x] + Sin[4*x]))/(96*a)
```

3.10.3 Rubi [A] (verified)

Time = 0.45 (sec) , antiderivative size = 67, normalized size of antiderivative = 1.02, number of steps used = 13, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.923, Rules used = {3042, 4306, 25, 3042, 4274, 3042, 3113, 2009, 3115, 3042, 3115, 24}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sin^4(x)}{a\csc(x)+a}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\csc(x)^4(a\csc(x)+a)}dx$$

$$\Big\downarrow \text{4306}$$

$$\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}-\frac{\int -((5a-4a\csc(x))\sin^4(x))dx}{a^2}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int (5a-4a\csc(x))\sin^4(x)dx}{a^2}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{5a-4a\csc(x)}{\csc(x)^4}dx}{a^2}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{4274}$$

$$\frac{5a\int \sin^4(x)dx-4a\int \sin^3(x)dx}{a^2}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{5a\int \sin(x)^4dx-4a\int \sin(x)^3dx}{a^2}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3113}$$

$$\frac{5a\int \sin(x)^4dx+4a\int \left(1-\cos^2(x)\right)d\cos(x)}{a^2}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{2009}$$

$$\frac{5a\int \sin(x)^4dx+4a\left(\cos(x)-\frac{\cos^3(x)}{3}\right)}{a^2}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3115}$$

$$\frac{5a\left(\frac{3}{4}\int \sin^2(x)dx-\frac{1}{4}\sin^3(x)\cos(x)\right)+4a\left(\cos(x)-\frac{\cos^3(x)}{3}\right)}{a^2}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{5a\left(\frac{3}{4}\int \sin(x)^2dx-\frac{1}{4}\sin^3(x)\cos(x)\right)+4a\left(\cos(x)-\frac{\cos^3(x)}{3}\right)}{a^2}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{3115}$$

$$\frac{5a\left(\frac{3}{4}\left(\frac{\int 1dx}{2}-\frac{1}{2}\sin(x)\cos(x)\right)-\frac{1}{4}\sin^3(x)\cos(x)\right)+4a\left(\cos(x)-\frac{\cos^3(x)}{3}\right)}{a^2}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

$$\Big\downarrow \text{24}$$

$$\frac{4a\left(\cos(x)-\frac{\cos^3(x)}{3}\right)+5a\left(\frac{3}{4}\left(\frac{x}{2}-\frac{1}{2}\sin(x)\cos(x)\right)-\frac{1}{4}\sin^3(x)\cos(x)\right)}{a^2}+\frac{\sin^3(x)\cos(x)}{a\csc(x)+a}$$

input
```
Int[Sin[x]^4/(a + a*Csc[x]),x]
```

output
```
(Cos[x]*Sin[x]^3)/(a + a*Csc[x]) + (4*a*(Cos[x] - Cos[x]^3/3) + 5*a*(-1/4*
(Cos[x]*Sin[x]^3) + (3*(x/2 - (Cos[x]*Sin[x])/2))/4))/a^2
```

3.10.3.1 Defintions of rubi rules used

rule 24
```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3113
```
Int[sin[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> Simp[-d^(-1)   Subst[Int[Exp
and[(1 - x^2)^((n - 1)/2), x], x], x, Cos[c + d*x]], x] /; FreeQ[{c, d}, x]
 && IGtQ[(n - 1)/2, 0]
```

rule 3115
```
Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :> Simp[(-b)*Cos[c + d*
x]*((b*Sin[c + d*x])^(n - 1)/(d*n)), x] + Simp[b^2*((n - 1)/n)   Int[(b*Sin
[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[
2*n]
```

rule 4274
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] :> Simp[a   Int[(d*Csc[e + f*x])^n, x], x] + Simp[b/d   In
t[(d*Csc[e + f*x])^(n + 1), x], x] /; FreeQ[{a, b, d, e, f, n}, x]
```

rule 4306
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[Cot[e + f*x]*((d*Csc[e + f*x])^n/(f*(a + b*Csc[e +
f*x]))), x] - Simp[1/a^2   Int[(d*Csc[e + f*x])^n*(a*(n - 1) - b*n*Csc[e +
f*x]), x], x] /; FreeQ[{a, b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && LtQ[n, 0
]
```

3.10.4 Maple [A] (verified)

Time = 0.59 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.68

method	result
parallelrisch	$\frac{3\cos(4x)\tan(x)-42\cos(2x)\tan(x)+168\cos(x)-8\cos(3x)-141\tan(x)+96\sec(x)+180x+104}{96a}$
risch	$\frac{15x}{8a}+\frac{7\,\mathrm{e}^{ix}}{8a}+\frac{7\,\mathrm{e}^{-ix}}{8a}+\frac{2}{(i+\mathrm{e}^{ix})a}+\frac{\sin(4x)}{32a}-\frac{\cos(3x)}{12a}-\frac{\sin(2x)}{2a}$
default	$\frac{\frac{64}{32\tan\left(\frac{x}{2}\right)+32}+\frac{2\left(\frac{7\tan\left(\frac{x}{2}\right)^{7}}{8}+\tan\left(\frac{x}{2}\right)^{6}+\frac{15\tan\left(\frac{x}{2}\right)^{5}}{8}+5\tan\left(\frac{x}{2}\right)^{4}-\frac{15\tan\left(\frac{x}{2}\right)^{3}}{8}+\frac{17\tan\left(\frac{x}{2}\right)^{2}}{3}-\frac{7\tan\left(\frac{x}{2}\right)}{8}+\frac{5}{3}\right)}{\left(1+\tan\left(\frac{x}{2}\right)^{2}\right)^{4}}+\frac{15\arctan\left(\tan\left(\frac{x}{2}\right)\right)}{4}}{a}$
norman	$\frac{\frac{15x}{8a}+\frac{15}{4a}+\frac{15x\tan\left(\frac{x}{2}\right)}{8a}+\frac{15x\tan\left(\frac{x}{2}\right)^{2}}{2a}+\frac{15x\tan\left(\frac{x}{2}\right)^{3}}{2a}+\frac{45x\tan\left(\frac{x}{2}\right)^{4}}{4a}+\frac{45x\tan\left(\frac{x}{2}\right)^{5}}{4a}+\frac{15x\tan\left(\frac{x}{2}\right)^{6}}{2a}+\frac{15x\tan\left(\frac{x}{2}\right)^{7}}{2a}+\frac{15x\tan\left(\frac{x}{2}\right)^{8}}{8a}+15\ldots}{\left(1+\tan\left(\frac{x}{2}\right)^{2}\right)^{4}(\tan\ldots}$

input
```
int(sin(x)^4/(a+a*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
1/96*(3*cos(4*x)*tan(x)-42*cos(2*x)*tan(x)+168*cos(x)-8*cos(3*x)-141*tan(x
)+96*sec(x)+180*x+104)/a
```

3.10.5 Fricas [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 81, normalized size of antiderivative = 1.23

$$\int \frac{\sin^4(x)}{a + a\csc(x)}\,dx = -\frac{6\cos(x)^5 + 8\cos(x)^4 - 25\cos(x)^3 - 45(x+1)\cos(x) - 48\cos(x)^2 - (6\cos(x)^4 - 2\cos(x)^3 - 27}{24(a\cos(x) + a\sin(x) + a)}$$

input
```
integrate(sin(x)^4/(a+a*csc(x)),x, algorithm="fricas")
```

output
```
-1/24*(6*cos(x)^5 + 8*cos(x)^4 - 25*cos(x)^3 - 45*(x + 1)*cos(x) - 48*cos(
x)^2 - (6*cos(x)^4 - 2*cos(x)^3 - 27*cos(x)^2 + 45*x + 21*cos(x) - 24)*sin
(x) - 45*x - 24)/(a*cos(x) + a*sin(x) + a)
```

3.10.6 Sympy [F]

$$\int \frac{\sin^4(x)}{a + a\csc(x)}\,dx = \frac{\int \frac{\sin^4(x)}{\csc(x)+1}\,dx}{a}$$

input
```
integrate(sin(x)**4/(a+a*csc(x)),x)
```

output
```
Integral(sin(x)**4/(csc(x) + 1), x)/a
```

3.10.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 230 vs. $2(58) = 116$.

Time = 0.33 (sec) , antiderivative size = 230, normalized size of antiderivative = 3.48

$$\int \frac{\sin^4(x)}{a + a\csc(x)}\,dx = \frac{\frac{19\sin(x)}{\cos(x)+1} + \frac{211\sin(x)^2}{(\cos(x)+1)^2} + \frac{91\sin(x)^3}{(\cos(x)+1)^3} + \frac{219\sin(x)^4}{(\cos(x)+1)^4} + \frac{165\sin(x)^5}{(\cos(x)+1)^5} + \frac{165\sin(x)^6}{(\cos(x)+1)^6} + \frac{45\sin(x)^7}{(\cos(x)+1)^7} + \frac{45\sin(x)^8}{(\cos(x)+1)^8} + \cdots}{12\left(a + \frac{a\sin(x)}{\cos(x)+1} + \frac{4a\sin(x)^2}{(\cos(x)+1)^2} + \frac{4a\sin(x)^3}{(\cos(x)+1)^3} + \frac{6a\sin(x)^4}{(\cos(x)+1)^4} + \frac{6a\sin(x)^5}{(\cos(x)+1)^5} + \frac{4a\sin(x)^6}{(\cos(x)+1)^6} + \frac{4a\sin(x)^7}{(\cos(x)+1)^7} + \frac{a\sin(x)^8}{(\cos(x)+1)^8} + \cdots\right.} + \frac{15\arctan\left(\frac{\sin(x)}{\cos(x)+1}\right)}{4a}$$

input
```
integrate(sin(x)^4/(a+a*csc(x)),x, algorithm="maxima")
```

output
```
1/12*(19*sin(x)/(cos(x) + 1) + 211*sin(x)^2/(cos(x) + 1)^2 + 91*sin(x)^3/(
cos(x) + 1)^3 + 219*sin(x)^4/(cos(x) + 1)^4 + 165*sin(x)^5/(cos(x) + 1)^5
+ 165*sin(x)^6/(cos(x) + 1)^6 + 45*sin(x)^7/(cos(x) + 1)^7 + 45*sin(x)^8/(
cos(x) + 1)^8 + 64)/(a + a*sin(x)/(cos(x) + 1) + 4*a*sin(x)^2/(cos(x) + 1)
^2 + 4*a*sin(x)^3/(cos(x) + 1)^3 + 6*a*sin(x)^4/(cos(x) + 1)^4 + 6*a*sin(x
)^5/(cos(x) + 1)^5 + 4*a*sin(x)^6/(cos(x) + 1)^6 + 4*a*sin(x)^7/(cos(x) +
1)^7 + a*sin(x)^8/(cos(x) + 1)^8 + a*sin(x)^9/(cos(x) + 1)^9) + 15/4*arcta
n(sin(x)/(cos(x) + 1))/a
```

3.10.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 91, normalized size of antiderivative = 1.38

$$\int \frac{\sin^4(x)}{a + a\csc(x)}\,dx = \frac{15x}{8a} + \frac{2}{a\left(\tan\left(\frac{1}{2}x\right) + 1\right)} + \frac{21\tan\left(\frac{1}{2}x\right)^7 + 24\tan\left(\frac{1}{2}x\right)^6 + 45\tan\left(\frac{1}{2}x\right)^5 + 120\tan\left(\frac{1}{2}x\right)^4 - 45\tan\left(\frac{1}{2}x\right)^3 + 136\tan\left(\frac{1}{2}x\right)^2 - 21\cdots}{12\left(\tan\left(\frac{1}{2}x\right)^2 + 1\right)^4 a}$$

input
```
integrate(sin(x)^4/(a+a*csc(x)),x, algorithm="giac")
```

output
```
15/8*x/a + 2/(a*(tan(1/2*x) + 1)) + 1/12*(21*tan(1/2*x)^7 + 24*tan(1/2*x)^
6 + 45*tan(1/2*x)^5 + 120*tan(1/2*x)^4 - 45*tan(1/2*x)^3 + 136*tan(1/2*x)^
2 - 21*tan(1/2*x) + 40)/((tan(1/2*x)^2 + 1)^4*a)
```

3.10.9 Mupad [B] (verification not implemented)

Time = 18.68 (sec) , antiderivative size = 93, normalized size of antiderivative = 1.41

$$\int \frac{\sin^4(x)}{a + a\csc(x)}\,dx = \frac{15\,x}{8\,a} + \frac{\frac{15\tan\left(\frac{x}{2}\right)^8}{4} + \frac{15\tan\left(\frac{x}{2}\right)^7}{4} + \frac{55\tan\left(\frac{x}{2}\right)^6}{4} + \frac{55\tan\left(\frac{x}{2}\right)^5}{4} + \frac{73\tan\left(\frac{x}{2}\right)^4}{4} + \frac{91\tan\left(\frac{x}{2}\right)^3}{12} + \frac{211\tan\left(\frac{x}{2}\right)^2}{12} + \frac{19\tan\left(\frac{x}{2}\right)}{12} + \frac{16}{3}}{a\left(\tan\left(\frac{x}{2}\right)^2 + 1\right)^4\left(\tan\left(\frac{x}{2}\right) + 1\right)}$$

input
```
int(sin(x)^4/(a + a/sin(x)),x)
```

output
```
(15*x)/(8*a) + ((19*tan(x/2))/12 + (211*tan(x/2)^2)/12 + (91*tan(x/2)^3)/1
2 + (73*tan(x/2)^4)/4 + (55*tan(x/2)^5)/4 + (55*tan(x/2)^6)/4 + (15*tan(x/
2)^7)/4 + (15*tan(x/2)^8)/4 + 16/3)/(a*(tan(x/2)^2 + 1)^4*(tan(x/2) + 1))
```

3.11 $\int \frac{1}{(a+a\csc(c+dx))^2}\,dx$

3.11.1 Optimal result

Integrand size = 12, antiderivative size = 57

$$\int \frac{1}{(a+a\csc(c+dx))^2}\,dx=\frac{x}{a^2}+\frac{4\cot(c+dx)}{3a^2d(1+\csc(c+dx))}+\frac{\cot(c+dx)}{3d(a+a\csc(c+dx))^2}$$

output
```
x/a^2+4/3*cot(d*x+c)/a^2/d/(1+csc(d*x+c))+1/3*cot(d*x+c)/d/(a+a*csc(d*x+c)
)^2
```

3.11.2 Mathematica [A] (verified)

Time = 0.60 (sec) , antiderivative size = 108, normalized size of antiderivative = 1.89

$$\int \frac{1}{(a+a\csc(c+dx))^2}\,dx$$
$$=\frac{3(-4+3c+3dx)\cos\left(\frac{1}{2}(c+dx)\right)+(10-3c-3dx)\cos\left(\frac{3}{2}(c+dx)\right)+6(-3+2c+2dx+(c+dx)\cos(c}{6a^2d\left(\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)\right)^3}$$

input
```
Integrate[(a + a*Csc[c + d*x])^(-2),x]
```

output
```
(3*(-4 + 3*c + 3*d*x)*Cos[(c + d*x)/2] + (10 - 3*c - 3*d*x)*Cos[(3*(c + d*
x))/2] + 6*(-3 + 2*c + 2*d*x + (c + d*x)*Cos[c + d*x])*Sin[(c + d*x)/2])/(
6*a^2*d*(Cos[(c + d*x)/2] + Sin[(c + d*x)/2])^3)
```

3.11.3 Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.07, number of steps used = 7, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.583, Rules used = {3042, 4264, 25, 3042, 4407, 3042, 4281}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{(a\csc(c+dx)+a)^2}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{(a\csc(c+dx)+a)^2}dx$$

$$\Big\downarrow \text{4264}$$

$$\frac{\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}-\frac{\int -\frac{3a-a\csc(c+dx)}{\csc(c+dx)a+a}dx}{3a^2}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int \frac{3a-a\csc(c+dx)}{\csc(c+dx)a+a}dx}{3a^2}+\frac{\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{3a-a\csc(c+dx)}{\csc(c+dx)a+a}dx}{3a^2}+\frac{\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}$$

$$\Big\downarrow \text{4407}$$

$$\frac{3x-4a\int \frac{\csc(c+dx)}{\csc(c+dx)a+a}dx}{3a^2}+\frac{\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}$$

$$\Big\downarrow \text{3042}$$

$$\frac{3x-4a\int \frac{\csc(c+dx)}{\csc(c+dx)a+a}dx}{3a^2}+\frac{\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}$$

$$\Big\downarrow \text{4281}$$

$$\frac{\frac{4a\cot(c+dx)}{d(a\csc(c+dx)+a)}+3x}{3a^2}+\frac{\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}$$

input
```
Int[(a + a*Csc[c + d*x])^(-2),x]
```

output
```
Cot[c + d*x]/(3*d*(a + a*Csc[c + d*x])^2) + (3*x + (4*a*Cot[c + d*x])/(d*(
a + a*Csc[c + d*x])))/(3*a^2)
```

3.11.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4264
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(-Cot[c
+ d*x])*((a + b*Csc[c + d*x])^n/(d*(2*n + 1))), x] + Simp[1/(a^2*(2*n + 1))
   Int[(a + b*Csc[c + d*x])^(n + 1)*(a*(2*n + 1) - b*(n + 1)*Csc[c + d*x]),
 x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && LeQ[n, -1] && Int
egerQ[2*n]
```

rule 4281
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[-Cot[e + f*x]/(f*(b + a*Csc[e + f*x])), x] /; FreeQ[{a, b, e, f}
, x] && EqQ[a^2 - b^2, 0]
```

rule 4407
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[c*(x/a), x] - Simp[(b*c - a*d)/a   Int[Csc[e + f*
x]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
- a*d, 0]
```

3.11.4 Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.45 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.95

method	result	size
risch	$\frac{x}{a^2}+\frac{6i e^{i(dx+c)}+4\,e^{2i(dx+c)}-\frac{10}{3}}{d\,a^2\left(i+e^{i(dx+c)}\right)^3}$	54
derivativedivides	$\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)-\frac{4}{3\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^3}+\frac{2}{\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^2}+\frac{8}{4\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+4}}{a^2 d}$	67
default	$\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)-\frac{4}{3\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^3}+\frac{2}{\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^2}+\frac{8}{4\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+4}}{a^2 d}$	67
parallelrisch	$\frac{(3dx-8)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^3+(9dx-18)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2+(9dx-6)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+3dx}{3d\,a^2\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^3}$	79
norman	$\frac{\frac{x}{a}+\frac{x\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^3}{a}-\frac{4\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2}{da}+\frac{3x\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{a}+\frac{3x\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2}{a}+\frac{2}{3ad}-\frac{2\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^3}{da}}{a\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)^3}$	118

input
```
int(1/(a+a*csc(d*x+c))^2,x,method=_RETURNVERBOSE)
```

output
```
x/a^2+2/3*(9*I*exp(I*(d*x+c))+6*exp(2*I*(d*x+c))-5)/d/a^2/(I+exp(I*(d*x+c)
))^3
```

3.11.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 124 vs. $2(53) = 106$.

Time = 0.25 (sec) , antiderivative size = 124, normalized size of antiderivative = 2.18

$$\int \frac{1}{(a+a\csc(c+dx))^2}\,dx$$
$$=\frac{(3\,dx-5)\cos(dx+c)^2-6\,dx-(3\,dx+4)\cos(dx+c)-(6\,dx+(3\,dx+5)\cos(dx+c)+1)\sin(dx+c}{3\left(a^2d\cos(dx+c)^2-a^2d\cos(dx+c)-2\,a^2d-(a^2d\cos(dx+c)+2\,a^2d)\sin(dx+c)\right)}$$

input
```
integrate(1/(a+a*csc(d*x+c))^2,x, algorithm="fricas")
```

output
```
1/3*((3*d*x - 5)*cos(d*x + c)^2 - 6*d*x - (3*d*x + 4)*cos(d*x + c) - (6*d*
x + (3*d*x + 5)*cos(d*x + c) + 1)*sin(d*x + c) + 1)/(a^2*d*cos(d*x + c)^2
- a^2*d*cos(d*x + c) - 2*a^2*d - (a^2*d*cos(d*x + c) + 2*a^2*d)*sin(d*x +
c))
```

3.11.6 Sympy [F]

$$\int \frac{1}{(a + a\csc(c + dx))^2}\,dx = \frac{\int \frac{1}{\csc^2(c+dx)+2\csc(c+dx)+1}\,dx}{a^2}$$

input
```
integrate(1/(a+a*csc(d*x+c))**2,x)
```

output
```
Integral(1/(csc(c + d*x)**2 + 2*csc(c + d*x) + 1), x)/a**2
```

3.11.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 142 vs. 2(53) = 106.

Time = 0.35 (sec) , antiderivative size = 142, normalized size of antiderivative = 2.49

$$\int \frac{1}{(a + a\csc(c + dx))^2}\,dx = \frac{2\left(\frac{\frac{9\sin(dx+c)}{\cos(dx+c)+1}+\frac{3\sin(dx+c)^2}{(\cos(dx+c)+1)^2}+4}{a^2+\frac{3a^2\sin(dx+c)}{\cos(dx+c)+1}+\frac{3a^2\sin(dx+c)^2}{(\cos(dx+c)+1)^2}+\frac{a^2\sin(dx+c)^3}{(\cos(dx+c)+1)^3}}+\frac{3\arctan\left(\frac{\sin(dx+c)}{\cos(dx+c)+1}\right)}{a^2}\right)}{3d}$$

input
```
integrate(1/(a+a*csc(d*x+c))^2,x, algorithm="maxima")
```

output
```
2/3*((9*sin(d*x + c)/(cos(d*x + c) + 1) + 3*sin(d*x + c)^2/(cos(d*x + c) +
 1)^2 + 4)/(a^2 + 3*a^2*sin(d*x + c)/(cos(d*x + c) + 1) + 3*a^2*sin(d*x +
c)^2/(cos(d*x + c) + 1)^2 + a^2*sin(d*x + c)^3/(cos(d*x + c) + 1)^3) + 3*a
rctan(sin(d*x + c)/(cos(d*x + c) + 1))/a^2)/d
```

3.11.8 Giac [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 60, normalized size of antiderivative = 1.05

$$\int \frac{1}{(a + a\csc(c + dx))^2}\,dx = \frac{\frac{3(dx+c)}{a^2}+\frac{2\left(3\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)^2+9\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)+4\right)}{a^2\left(\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)+1\right)^3}}{3d}$$

input
```
integrate(1/(a+a*csc(d*x+c))^2,x, algorithm="giac")
```

output
```
1/3*(3*(d*x + c)/a^2 + 2*(3*tan(1/2*d*x + 1/2*c)^2 + 9*tan(1/2*d*x + 1/2*c
) + 4)/(a^2*(tan(1/2*d*x + 1/2*c) + 1)^3))/d
```

3.11.9 Mupad [B] (verification not implemented)

Time = 18.08 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.91

$$\int \frac{1}{(a+a\csc(c+dx))^2}\,dx=\frac{x}{a^2}+\frac{2\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)^2+6\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)+\frac{8}{3}}{a^2\,d\,\left(\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)+1\right)^3}$$

input
```
int(1/(a + a/sin(c + d*x))^2,x)
```

output
```
x/a^2 + (6*tan(c/2 + (d*x)/2) + 2*tan(c/2 + (d*x)/2)^2 + 8/3)/(a^2*d*(tan(
c/2 + (d*x)/2) + 1)^3)
```

3.12 $\int \frac{1}{(a+a\csc(c+dx))^3}\,dx$

3.12.1 Optimal result

Integrand size = 12, antiderivative size = 88

$$\int \frac{1}{(a+a\csc(c+dx))^3}\,dx=\frac{x}{a^3}+\frac{\cot(c+dx)}{5d(a+a\csc(c+dx))^3}+\frac{7\cot(c+dx)}{15ad(a+a\csc(c+dx))^2}+\frac{22\cot(c+dx)}{15d\,(a^3+a^3\csc(c+dx))}$$

output
```
x/a^3+1/5*cot(d*x+c)/d/(a+a*csc(d*x+c))^3+7/15*cot(d*x+c)/a/d/(a+a*csc(d*x
+c))^2+22/15*cot(d*x+c)/d/(a^3+a^3*csc(d*x+c))
```

3.12.2 Mathematica [A] (verified)

Time = 1.05 (sec) , antiderivative size = 123, normalized size of antiderivative = 1.40

$$\int \frac{1}{(a+a\csc(c+dx))^3}\,dx = \frac{15c+15dx+\frac{3}{\left(\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)\right)^4}-\frac{13}{\left(\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)\right)^2}+\frac{2\sin\left(\frac{1}{2}(c+dx)\right)(-38+16\cos(2(c+dx))-51\sin(c+}{\left(\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)\right)^5}}{15a^3d}$$

input
```
Integrate[(a + a*Csc[c + d*x])^(-3),x]
```

output
```
(15*c + 15*d*x + 3/(Cos[(c + d*x)/2] + Sin[(c + d*x)/2])^4 - 13/(Cos[(c +
d*x)/2] + Sin[(c + d*x)/2])^2 + (2*Sin[(c + d*x)/2]*(-38 + 16*Cos[2*(c + d
*x)] - 51*Sin[c + d*x]))/(Cos[(c + d*x)/2] + Sin[(c + d*x)/2])^5)/(15*a^3*
d)
```

3.12.3 Rubi [A] (verified)

Time = 0.56 (sec) , antiderivative size = 98, normalized size of antiderivative = 1.11, number of steps used = 10, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}} = 0.833$, Rules used = {3042, 4264, 25, 3042, 4410, 25, 3042, 4407, 3042, 4281}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{(a\csc(c+dx)+a)^3}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{(a\csc(c+dx)+a)^3}dx$$

$$\Big\downarrow \text{4264}$$

$$\frac{\cot(c+dx)}{5d(a\csc(c+dx)+a)^3}-\frac{\int -\frac{5a-2a\csc(c+dx)}{(\csc(c+dx)a+a)^2}dx}{5a^2}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int \frac{5a-2a\csc(c+dx)}{(\csc(c+dx)a+a)^2}dx}{5a^2}+\frac{\cot(c+dx)}{5d(a\csc(c+dx)+a)^3}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{5a-2a\csc(c+dx)}{(\csc(c+dx)a+a)^2}dx}{5a^2}+\frac{\cot(c+dx)}{5d(a\csc(c+dx)+a)^3}$$

$$\Big\downarrow \text{4410}$$

$$\frac{\frac{7a\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}-\frac{\int -\frac{15a^2-7a^2\csc(c+dx)}{\csc(c+dx)a+a}dx}{3a^2}}{5a^2}+\frac{\cot(c+dx)}{5d(a\csc(c+dx)+a)^3}$$

$$\Big\downarrow \text{25}$$

$$\frac{\frac{\int \frac{15a^2-7a^2\csc(c+dx)}{\csc(c+dx)a+a}dx}{3a^2}+\frac{7a\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}}{5a^2}+\frac{\cot(c+dx)}{5d(a\csc(c+dx)+a)^3}$$

$$\Big\downarrow \ 3042$$

$$\frac{\frac{\int \frac{15a^2-7a^2\csc(c+dx)}{\csc(c+dx)a+a}dx}{3a^2}+\frac{7a\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}}{5a^2}+\frac{\cot(c+dx)}{5d(a\csc(c+dx)+a)^3}$$

$$\Big\downarrow \ 4407$$

$$\frac{\frac{15ax-22a^2\int \frac{\csc(c+dx)}{\csc(c+dx)a+a}dx}{3a^2}+\frac{7a\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}}{5a^2}+\frac{\cot(c+dx)}{5d(a\csc(c+dx)+a)^3}$$

$$\Big\downarrow \ 3042$$

$$\frac{\frac{15ax-22a^2\int \frac{\csc(c+dx)}{\csc(c+dx)a+a}dx}{3a^2}+\frac{7a\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}}{5a^2}+\frac{\cot(c+dx)}{5d(a\csc(c+dx)+a)^3}$$

$$\Big\downarrow \ 4281$$

$$\frac{\frac{\frac{22a^2\cot(c+dx)}{d(a\csc(c+dx)+a)}+15ax}{3a^2}+\frac{7a\cot(c+dx)}{3d(a\csc(c+dx)+a)^2}}{5a^2}+\frac{\cot(c+dx)}{5d(a\csc(c+dx)+a)^3}$$

input
```
Int[(a + a*Csc[c + d*x])^(-3),x]
```

output
```
Cot[c + d*x]/(5*d*(a + a*Csc[c + d*x])^3) + ((7*a*Cot[c + d*x])/(3*d*(a +
a*Csc[c + d*x])^2) + (15*a*x + (22*a^2*Cot[c + d*x])/(d*(a + a*Csc[c + d*x
])))/(3*a^2))/(5*a^2)
```

3.12.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4264
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(-Cot[c
+ d*x])*((a + b*Csc[c + d*x])^n/(d*(2*n + 1))), x] + Simp[1/(a^2*(2*n + 1))
   Int[(a + b*Csc[c + d*x])^(n + 1)*(a*(2*n + 1) - b*(n + 1)*Csc[c + d*x]),
 x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && LeQ[n, -1] && Int
egerQ[2*n]
```

rule 4281
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[-Cot[e + f*x]/(f*(b + a*Csc[e + f*x])), x] /; FreeQ[{a, b, e, f}
, x] && EqQ[a^2 - b^2, 0]
```

rule 4407
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[c*(x/a), x] - Simp[(b*c - a*d)/a   Int[Csc[e + f*
x]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
- a*d, 0]
```

rule 4410
```
Int[(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_)*(csc[(e_.) + (f_.)*(x_)]*(d
_.) + (c_)), x_Symbol] :> Simp[(-(b*c - a*d))*Cot[e + f*x]*((a + b*Csc[e +
f*x])^m/(b*f*(2*m + 1))), x] + Simp[1/(a^2*(2*m + 1))   Int[(a + b*Csc[e +
f*x])^(m + 1)*Simp[a*c*(2*m + 1) - (b*c - a*d)*(m + 1)*Csc[e + f*x], x], x]
, x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && LtQ[m, -1] &&
EqQ[a^2 - b^2, 0] && IntegerQ[2*m]
```

3.12.4 Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.58 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.88

method	result
risch	$\frac{x}{a^{3}}+\frac{-\frac{74\,{\mathrm e}^{2 i (d x +c)}}{3}+18 i {\mathrm e}^{3 i (d x +c)}-\frac{46 i {\mathrm e}^{i (d x +c)}}{3}+6\,{\mathrm e}^{4 i (d x +c)}+\frac{64}{15}}{d\,a^{3} \left(i+{\mathrm e}^{i (d x +c)}\right)^{5}}$
derivativedivides	$\frac{2 \arctan \left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)\right)-\frac{4}{\left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)+1\right)^{4}}+\frac{8}{5 \left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)+1\right)^{5}}+\frac{4}{3 \left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)+1\right)^{3}}+\frac{2}{\left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)+1\right)^{2}}+\frac{16}{8 \tan \left(\frac{d x}{2}+\frac{c}{2}\right.}}{d\,a^{3}}$
default	$\frac{2 \arctan \left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)\right)-\frac{4}{\left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)+1\right)^{4}}+\frac{8}{5 \left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)+1\right)^{5}}+\frac{4}{3 \left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)+1\right)^{3}}+\frac{2}{\left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)+1\right)^{2}}+\frac{16}{8 \tan \left(\frac{d x}{2}+\frac{c}{2}\right.}}{d\,a^{3}}$
parallelrisch	$\frac{\left(15 d x -38\right) \tan \left(\frac{d x}{2}+\frac{c}{2}\right)^{5}+\left(75 d x -160\right) \tan \left(\frac{d x}{2}+\frac{c}{2}\right)^{4}+\left(150 d x -230\right) \tan \left(\frac{d x}{2}+\frac{c}{2}\right)^{3}+\left(150 d x -90\right) \tan \left(\frac{d x}{2}+\frac{c}{2}\right)^{2}+75 \tan}{15 d\,a^{3} \left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)+1\right)^{5}}$
norman	$\frac{\frac{x}{a}+\frac{x \tan \left(\frac{d x}{2}+\frac{c}{2}\right)^{5}}{a}+\frac{2 \tan \left(\frac{d x}{2}+\frac{c}{2}\right)^{4}}{d a}+\frac{5 x \tan \left(\frac{d x}{2}+\frac{c}{2}\right)}{a}+\frac{10 x \tan \left(\frac{d x}{2}+\frac{c}{2}\right)^{2}}{a}+\frac{10 x \tan \left(\frac{d x}{2}+\frac{c}{2}\right)^{3}}{a}+\frac{5 x \tan \left(\frac{d x}{2}+\frac{c}{2}\right)^{4}}{a}+\frac{44}{15 a d}+\frac{10 \mathrm{ta}}{}}{a^{2} \left(\tan \left(\frac{d x}{2}+\frac{c}{2}\right)+1\right)^{5}}$

input
```
int(1/(a+a*csc(d*x+c))^3,x,method=_RETURNVERBOSE)
```

output
```
x/a^3+2/15*(-185*exp(2*I*(d*x+c))+135*I*exp(3*I*(d*x+c))-115*I*exp(I*(d*x+
c))+45*exp(4*I*(d*x+c))+32)/d/a^3/(I+exp(I*(d*x+c)))^5
```

3.12.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 181 vs. $2(82) = 164$.

Time = 0.25 (sec) , antiderivative size = 181, normalized size of antiderivative = 2.06

$$\int \frac{1}{(a + a\csc(c + dx))^3}\,dx = \frac{(15\,dx + 32)\cos{(dx + c)}^3 + (45\,dx - 19)\cos{(dx + c)}^2 - 60\,dx - 6\,(5\,dx + 9)\cos{(dx + c)} + \left((15\,dx - 32\right.}{15\,\left(a^3d\cos{(dx + c)}^3 + 3\,a^3d\cos{(dx + c)}^2 - 2\,a^3d\cos{(dx + c)} - 4\,a^3d + \left(a^3d\cos{(\right.\right.}$$

input
```
integrate(1/(a+a*csc(d*x+c))^3,x, algorithm="fricas")
```

output
```
1/15*((15*d*x + 32)*cos(d*x + c)^3 + (45*d*x - 19)*cos(d*x + c)^2 - 60*d*x
 - 6*(5*d*x + 9)*cos(d*x + c) + ((15*d*x - 32)*cos(d*x + c)^2 - 60*d*x - 3
*(10*d*x + 17)*cos(d*x + c) + 3)*sin(d*x + c) - 3)/(a^3*d*cos(d*x + c)^3 +
 3*a^3*d*cos(d*x + c)^2 - 2*a^3*d*cos(d*x + c) - 4*a^3*d + (a^3*d*cos(d*x
+ c)^2 - 2*a^3*d*cos(d*x + c) - 4*a^3*d)*sin(d*x + c))
```

3.12.6 Sympy [F]

$$\int \frac{1}{(a + a\csc(c + dx))^3}\,dx = \frac{\int \frac{1}{\csc^3{(c+dx)} + 3\csc^2{(c+dx)} + 3\csc{(c+dx)} + 1}\,dx}{a^3}$$

input
```
integrate(1/(a+a*csc(d*x+c))**3,x)
```

output
```
Integral(1/(csc(c + d*x)**3 + 3*csc(c + d*x)**2 + 3*csc(c + d*x) + 1), x)/
a**3
```

3.12.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 228 vs. 2(82) = 164.

Time = 0.32 (sec) , antiderivative size = 228, normalized size of antiderivative = 2.59

$$\int \frac{1}{(a+a\csc(c+dx))^3}\,dx = \frac{2\left(\frac{\frac{95\sin(dx+c)}{\cos(dx+c)+1}+\frac{145\sin(dx+c)^2}{(\cos(dx+c)+1)^2}+\frac{75\sin(dx+c)^3}{(\cos(dx+c)+1)^3}+\frac{15\sin(dx+c)^4}{(\cos(dx+c)+1)^4}+22}{a^3+\frac{5a^3\sin(dx+c)}{\cos(dx+c)+1}+\frac{10a^3\sin(dx+c)^2}{(\cos(dx+c)+1)^2}+\frac{10a^3\sin(dx+c)^3}{(\cos(dx+c)+1)^3}+\frac{5a^3\sin(dx+c)^4}{(\cos(dx+c)+1)^4}+\frac{a^3\sin(dx+c)^5}{(\cos(dx+c)+1)^5}}+\frac{15\arctan\left(\frac{\sin(dx+c)}{\cos(dx+c)+1}\right)}{a^3}\right)}{15d}$$

input
```
integrate(1/(a+a*csc(d*x+c))^3,x, algorithm="maxima")
```

output
```
2/15*((95*sin(d*x + c)/(cos(d*x + c) + 1) + 145*sin(d*x + c)^2/(cos(d*x +
c) + 1)^2 + 75*sin(d*x + c)^3/(cos(d*x + c) + 1)^3 + 15*sin(d*x + c)^4/(co
s(d*x + c) + 1)^4 + 22)/(a^3 + 5*a^3*sin(d*x + c)/(cos(d*x + c) + 1) + 10*
a^3*sin(d*x + c)^2/(cos(d*x + c) + 1)^2 + 10*a^3*sin(d*x + c)^3/(cos(d*x +
 c) + 1)^3 + 5*a^3*sin(d*x + c)^4/(cos(d*x + c) + 1)^4 + a^3*sin(d*x + c)^
5/(cos(d*x + c) + 1)^5) + 15*arctan(sin(d*x + c)/(cos(d*x + c) + 1))/a^3)/
d
```

3.12.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 86, normalized size of antiderivative = 0.98

$$\int \frac{1}{(a+a\csc(c+dx))^3}\,dx = \frac{\frac{15(dx+c)}{a^3}+\frac{2\left(15\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)^4+75\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)^3+145\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)^2+95\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)+22\right)}{a^3\left(\tan\left(\frac{1}{2}dx+\frac{1}{2}c\right)+1\right)^5}}{15d}$$

input
```
integrate(1/(a+a*csc(d*x+c))^3,x, algorithm="giac")
```

output
```
1/15*(15*(d*x + c)/a^3 + 2*(15*tan(1/2*d*x + 1/2*c)^4 + 75*tan(1/2*d*x + 1
/2*c)^3 + 145*tan(1/2*d*x + 1/2*c)^2 + 95*tan(1/2*d*x + 1/2*c) + 22)/(a^3*
(tan(1/2*d*x + 1/2*c) + 1)^5))/d
```

3.12.9 Mupad [B] (verification not implemented)

Time = 19.65 (sec) , antiderivative size = 78, normalized size of antiderivative = 0.89

$$\int \frac{1}{(a+a\csc(c+dx))^3}\,dx$$
$$=\frac{x}{a^3}+\frac{2\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)^4+10\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)^3+\frac{58\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)^2}{3}+\frac{38\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)}{3}+\frac{44}{15}}{a^3\,d\,\left(\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)+1\right)^5}$$

input
```
int(1/(a + a/sin(c + d*x))^3,x)
```

output
```
x/a^3 + ((38*tan(c/2 + (d*x)/2))/3 + (58*tan(c/2 + (d*x)/2)^2)/3 + 10*tan(
c/2 + (d*x)/2)^3 + 2*tan(c/2 + (d*x)/2)^4 + 44/15)/(a^3*d*(tan(c/2 + (d*x)
/2) + 1)^5)
```

3.13 $\int (a+a\csc(x))^{5/2}\,dx$

3.13.1 Optimal result

Integrand size = 10, antiderivative size = 65

$$\int (a+a\csc(x))^{5/2}\,dx = -2a^{5/2}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right)-\frac{14a^3\cot(x)}{3\sqrt{a+a\csc(x)}}-\frac{2}{3}a^2\cot(x)\sqrt{a+a\csc(x)}$$

output
```
-2*a^(5/2)*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))-14/3*a^3*cot(x)/(a+a*
csc(x))^(1/2)-2/3*a^2*cot(x)*(a+a*csc(x))^(1/2)
```

3.13.2 Mathematica [A] (verified)

Time = 2.23 (sec) , antiderivative size = 80, normalized size of antiderivative = 1.23

$$\int (a+a\csc(x))^{5/2}\,dx = -\frac{2a^2\sqrt{a(1+\csc(x))}\left(3\arctan\left(\sqrt{-1+\csc(x)}\right)+\sqrt{-1+\csc(x)}(8+\csc(x))\right)\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)}{3\sqrt{-1+\csc(x)}\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)}$$

input
```
Integrate[(a + a*Csc[x])^(5/2),x]
```

output
```
(-2*a^2*Sqrt[a*(1 + Csc[x])]*(3*ArcTan[Sqrt[-1 + Csc[x]]] + Sqrt[-1 + Csc[
x]]*(8 + Csc[x]))*(Cos[x/2] - Sin[x/2]))/(3*Sqrt[-1 + Csc[x]]*(Cos[x/2] +
Sin[x/2]))
```

3.13.3 Rubi [A] (verified)

Time = 0.45 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.06, number of steps used = 10, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.900, Rules used = {3042, 4262, 27, 3042, 4403, 3042, 4261, 216, 4279}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (a\csc(x)+a)^{5/2}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int (a\csc(x)+a)^{5/2}dx$$

$$\Big\downarrow \text{4262}$$

$$\frac{2}{3}a\int \frac{1}{2}\sqrt{\csc(x)a+a}(7\csc(x)a+3a)dx-\frac{2}{3}a^2\cot(x)\sqrt{a\csc(x)+a}$$

$$\Big\downarrow \text{27}$$

$$\frac{1}{3}a\int \sqrt{\csc(x)a+a}(7\csc(x)a+3a)dx-\frac{2}{3}a^2\cot(x)\sqrt{a\csc(x)+a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{1}{3}a\int \sqrt{\csc(x)a+a}(7\csc(x)a+3a)dx-\frac{2}{3}a^2\cot(x)\sqrt{a\csc(x)+a}$$

$$\Big\downarrow \text{4403}$$

$$\frac{1}{3}a\left(3a\int \sqrt{\csc(x)a+a}dx+7a\int \csc(x)\sqrt{\csc(x)a+a}dx\right)-\frac{2}{3}a^2\cot(x)\sqrt{a\csc(x)+a}$$

$$\Big\downarrow \text{3042}$$

$$\frac{1}{3}a\left(3a\int \sqrt{\csc(x)a+a}dx+7a\int \csc(x)\sqrt{\csc(x)a+a}dx\right)-\frac{2}{3}a^2\cot(x)\sqrt{a\csc(x)+a}$$

$$\Big\downarrow \text{4261}$$

$$\frac{1}{3}a\left(7a\int \csc(x)\sqrt{\csc(x)a+a}dx-6a^2\int \frac{1}{\frac{a^2\cot^2(x)}{\csc(x)a+a}+a}d\frac{a\cot(x)}{\sqrt{\csc(x)a+a}}\right)-\frac{2}{3}a^2\cot(x)\sqrt{a\csc(x)+a}$$

$$\Big\downarrow \text{216}$$

$$\frac{1}{3}a\left(7a\int \csc(x)\sqrt{\csc(x)a+a}dx-6a^{3/2}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)\right)-\frac{2}{3}a^2\cot(x)\sqrt{a\csc(x)+a}$$

$$\Big\downarrow \text{4279}$$

$$\frac{1}{3}a\left(-6a^{3/2}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)-\frac{14a^2\cot(x)}{\sqrt{a\csc(x)+a}}\right)-\frac{2}{3}a^2\cot(x)\sqrt{a\csc(x)+a}$$

input
```
Int[(a + a*Csc[x])^(5/2),x]
```

output
```
(-2*a^2*Cot[x]*Sqrt[a + a*Csc[x]])/3 + (a*(-6*a^(3/2)*ArcTan[(Sqrt[a]*Cot[
x])/Sqrt[a + a*Csc[x]]] - (14*a^2*Cot[x])/Sqrt[a + a*Csc[x]]))/3
```

3.13.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 216
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[b, 2]))*A
rcTan[Rt[b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4261
```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Simp[-2*(b/d)
  Subst[Int[1/(a + x^2), x], x, b*(Cot[c + d*x]/Sqrt[a + b*Csc[c + d*x]])],
 x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

rule 4262
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(-b^2)*C
ot[c + d*x]*((a + b*Csc[c + d*x])^(n - 2)/(d*(n - 1))), x] + Simp[a/(n - 1)
   Int[(a + b*Csc[c + d*x])^(n - 2)*(a*(n - 1) + b*(3*n - 4)*Csc[c + d*x]),
 x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && GtQ[n, 1] && Inte
gerQ[2*n]
```

rule 4279
```
Int[csc[(e_.) + (f_.)*(x_)]*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)], x_S
ymbol] :> Simp[-2*b*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x]])), x] /; Free
Q[{a, b, e, f}, x] && EqQ[a^2 - b^2, 0]
```

rule 4403
```
Int[Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)]*(csc[(e_.) + (f_.)*(x_)]*(d_
.) + (c_)), x_Symbol] :> Simp[c   Int[Sqrt[a + b*Csc[e + f*x]], x], x] + Si
mp[d   Int[Sqrt[a + b*Csc[e + f*x]]*Csc[e + f*x], x], x] /; FreeQ[{a, b, c,
 d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

3.13.4 Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 274 vs. $2(51) = 102$.

Time = 0.88 (sec) , antiderivative size = 275, normalized size of antiderivative = 4.23

method	result
default	$\frac{\csc(x)\left(\frac{a\left(\csc(x)(1-\cos(x))^2+2-2\cos(x)+\sin(x)\right)}{1-\cos(x)}\right)^{\frac{5}{2}}(1-\cos(x))\left(2\csc(x)^3(1-\cos(x))^3+3\sqrt{2}\,(\csc(x)-\cot(x))^{\frac{3}{2}}\ln\left(-\frac{\csc(x)-\cot(x)+\sqrt{c}\cdots}{\sqrt{\csc(x)-\cot(x)}\,\sqrt{2}\cdots}\right.\right.}{\cdots}$

input
```
int((a+a*csc(x))^(5/2),x,method=_RETURNVERBOSE)
```

output
```
1/12*csc(x)*(a/(1-cos(x))*(csc(x)*(1-cos(x))^2+2-2*cos(x)+sin(x)))^(5/2)/(
csc(x)-cot(x)+1)^5*(1-cos(x))*(2*csc(x)^3*(1-cos(x))^3+3*2^(1/2)*(csc(x)-c
ot(x))^(3/2)*ln(-(csc(x)-cot(x)+(csc(x)-cot(x))^(1/2)*2^(1/2)+1)/((csc(x)-
cot(x))^(1/2)*2^(1/2)-csc(x)+cot(x)-1))+12*2^(1/2)*(csc(x)-cot(x))^(3/2)*a
rctan((csc(x)-cot(x))^(1/2)*2^(1/2)+1)+12*2^(1/2)*(csc(x)-cot(x))^(3/2)*ar
ctan((csc(x)-cot(x))^(1/2)*2^(1/2)-1)+3*2^(1/2)*(csc(x)-cot(x))^(3/2)*ln(-
((csc(x)-cot(x))^(1/2)*2^(1/2)-csc(x)+cot(x)-1)/(csc(x)-cot(x)+(csc(x)-cot
(x))^(1/2)*2^(1/2)+1))+30*csc(x)^2*(1-cos(x))^2-30*csc(x)+30*cot(x)-2)*2^(
1/2)
```

3.13.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 142 vs. 2(51) = 102.

Time = 0.27 (sec) , antiderivative size = 318, normalized size of antiderivative = 4.89

$$\int (a + a\csc(x))^{5/2}\,dx = \left[\frac{3\left(a^2\cos(x)^2 - a^2 - (a^2\cos(x) + a^2)\sin(x)\right)\sqrt{-a}\log\left(\frac{2\,a\cos(x)^2 - 2\left(\cos(x)^2 + (\cos(x)+1)\sin(x)\right.}{\text{c}}\right.}{3\,(}\right.$$

input
```
integrate((a+a*csc(x))^(5/2),x, algorithm="fricas")
```

output
```
[1/3*(3*(a^2*cos(x)^2 - a^2 - (a^2*cos(x) + a^2)*sin(x))*sqrt(-a)*log((2*a
*cos(x)^2 - 2*(cos(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt(-a)*sqrt((a*sin(x)
 + a)/sin(x)) + a*cos(x) - (2*a*cos(x) + a)*sin(x) - a)/(cos(x) + sin(x) +
 1)) + 2*(8*a^2*cos(x)^2 + a^2*cos(x) - 7*a^2 + (8*a^2*cos(x) + 7*a^2)*sin
(x))*sqrt((a*sin(x) + a)/sin(x)))/(cos(x)^2 - (cos(x) + 1)*sin(x) - 1), 2/
3*(3*(a^2*cos(x)^2 - a^2 - (a^2*cos(x) + a^2)*sin(x))*sqrt(a)*arctan(-sqrt
(a)*sqrt((a*sin(x) + a)/sin(x))*(cos(x) - sin(x) + 1)/(a*cos(x) + a*sin(x)
 + a)) + (8*a^2*cos(x)^2 + a^2*cos(x) - 7*a^2 + (8*a^2*cos(x) + 7*a^2)*sin
(x))*sqrt((a*sin(x) + a)/sin(x)))/(cos(x)^2 - (cos(x) + 1)*sin(x) - 1)]
```

3.13.6 Sympy [F]

$$\int (a + a\csc(x))^{5/2}\,dx = \int (a\csc(x) + a)^{\frac{5}{2}}\,dx$$

input
```
integrate((a+a*csc(x))**(5/2),x)
```

output
```
Integral((a*csc(x) + a)**(5/2), x)
```

3.13.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 417 vs. $2(51) = 102$.

Time = 0.35 (sec) , antiderivative size = 417, normalized size of antiderivative = 6.42

$$\int (a+a\csc(x))^{5/2}\,dx = \frac{1}{22}\sqrt{2}a^{\frac{5}{2}}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{11}{2}} + \frac{5}{18}\sqrt{2}a^{\frac{5}{2}}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{9}{2}} + \frac{9}{14}\sqrt{2}a^{\frac{5}{2}}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{7}{2}} + \frac{1}{2}\sqrt{2}a^{\frac{5}{2}}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{5}{2}} - \frac{2}{3}\sqrt{2}a^{\frac{5}{2}}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{3}{2}} + \sqrt{2}\left(\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\sqrt{2}+2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right) + \sqrt{2}\arctan\left(-\frac{1}{2}\sqrt{2}\left(\sqrt{2}-2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right)\right)a^{\frac{5}{2}} - 2\sqrt{2}a^{\frac{5}{2}}\sqrt{\frac{\sin(x)}{\cos(x)+1}} - \frac{\frac{693\sqrt{2}a^{\frac{5}{2}}\sin(x)}{\cos(x)+1} + \frac{1155\sqrt{2}a^{\frac{5}{2}}\sin(x)^2}{(\cos(x)+1)^2} + \frac{1386\sqrt{2}a^{\frac{5}{2}}\sin(x)^3}{(\cos(x)+1)^3} + \frac{990\sqrt{2}a^{\frac{5}{2}}\sin(x)^4}{(\cos(x)+1)^4} + \frac{385\sqrt{2}a^{\frac{5}{2}}\sin(x)^5}{(\cos(x)+1)^5} + \frac{63\sqrt{2}a^{\frac{5}{2}}\sin(x)^6}{(\cos(x)+1)^6}}{1386\sqrt{\frac{\sin(x)}{\cos(x)+1}}} - \frac{\frac{7\sqrt{2}a^{\frac{5}{2}}\sin(x)}{\cos(x)+1} + \frac{105\sqrt{2}a^{\frac{5}{2}}\sin(x)^2}{(\cos(x)+1)^2} - \frac{210\sqrt{2}a^{\frac{5}{2}}\sin(x)^3}{(\cos(x)+1)^3} - \frac{70\sqrt{2}a^{\frac{5}{2}}\sin(x)^4}{(\cos(x)+1)^4} - \frac{21\sqrt{2}a^{\frac{5}{2}}\sin(x)^5}{(\cos(x)+1)^5} - \frac{3\sqrt{2}a^{\frac{5}{2}}\sin(x)^6}{(\cos(x)+1)^6}}{42\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{5}{2}}}$$

input
```
integrate((a+a*csc(x))^(5/2),x, algorithm="maxima")
```

output
```
1/22*sqrt(2)*a^(5/2)*(sin(x)/(cos(x) + 1))^(11/2) + 5/18*sqrt(2)*a^(5/2)*(
sin(x)/(cos(x) + 1))^(9/2) + 9/14*sqrt(2)*a^(5/2)*(sin(x)/(cos(x) + 1))^(7
/2) + 1/2*sqrt(2)*a^(5/2)*(sin(x)/(cos(x) + 1))^(5/2) - 2/3*sqrt(2)*a^(5/2
)*(sin(x)/(cos(x) + 1))^(3/2) + sqrt(2)*(sqrt(2)*arctan(1/2*sqrt(2)*(sqrt(
2) + 2*sqrt(sin(x)/(cos(x) + 1)))) + sqrt(2)*arctan(-1/2*sqrt(2)*(sqrt(2)
- 2*sqrt(sin(x)/(cos(x) + 1)))))*a^(5/2) - 2*sqrt(2)*a^(5/2)*sqrt(sin(x)/(
cos(x) + 1)) - 1/1386*(693*sqrt(2)*a^(5/2)*sin(x)/(cos(x) + 1) + 1155*sqrt
(2)*a^(5/2)*sin(x)^2/(cos(x) + 1)^2 + 1386*sqrt(2)*a^(5/2)*sin(x)^3/(cos(x
) + 1)^3 + 990*sqrt(2)*a^(5/2)*sin(x)^4/(cos(x) + 1)^4 + 385*sqrt(2)*a^(5/
2)*sin(x)^5/(cos(x) + 1)^5 + 63*sqrt(2)*a^(5/2)*sin(x)^6/(cos(x) + 1)^6)/s
qrt(sin(x)/(cos(x) + 1)) - 1/42*(7*sqrt(2)*a^(5/2)*sin(x)/(cos(x) + 1) + 1
05*sqrt(2)*a^(5/2)*sin(x)^2/(cos(x) + 1)^2 - 210*sqrt(2)*a^(5/2)*sin(x)^3/
(cos(x) + 1)^3 - 70*sqrt(2)*a^(5/2)*sin(x)^4/(cos(x) + 1)^4 - 21*sqrt(2)*a
^(5/2)*sin(x)^5/(cos(x) + 1)^5 - 3*sqrt(2)*a^(5/2)*sin(x)^6/(cos(x) + 1)^6
)/(sin(x)/(cos(x) + 1))^(5/2)
```

3.13.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 250 vs. $2(51) = 102$.

Time = 0.46 (sec) , antiderivative size = 250, normalized size of antiderivative = 3.85

$$\int (a+a\csc(x))^{5/2}\,dx = \left(a^2\sqrt{|a|}+a|a|^{\frac{3}{2}}\right)\arctan\left(\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|}+2\sqrt{a\tan\left(\frac{1}{2}x\right)}\right)}{2\sqrt{|a|}}\right) + \left(a^2\sqrt{|a|}+a|a|^{\frac{3}{2}}\right)\arctan\left(-\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|}-2\sqrt{a\tan\left(\frac{1}{2}x\right)}\right)}{2\sqrt{|a|}}\right) + \frac{1}{2}\left(a^2\sqrt{|a|}-a|a|^{\frac{3}{2}}\right)\log\left(a\tan\left(\frac{1}{2}x\right)+\sqrt{2}\sqrt{a\tan\left(\frac{1}{2}x\right)}\sqrt{|a|}+|a|\right) - \frac{1}{2}\left(a^2\sqrt{|a|}-a|a|^{\frac{3}{2}}\right)\log\left(a\tan\left(\frac{1}{2}x\right)-\sqrt{2}\sqrt{a\tan\left(\frac{1}{2}x\right)}\sqrt{|a|}+|a|\right) + \frac{1}{6}\sqrt{2}\left(\sqrt{a\tan\left(\frac{1}{2}x\right)}a^2\tan\left(\frac{1}{2}x\right)+15\sqrt{a\tan\left(\frac{1}{2}x\right)}a^2\right) - \frac{\sqrt{2}\left(15a^4\tan\left(\frac{1}{2}x\right)+a^4\right)}{6\sqrt{a\tan\left(\frac{1}{2}x\right)}a\tan\left(\frac{1}{2}x\right)}$$

input
```
integrate((a+a*csc(x))^(5/2),x, algorithm="giac")
```

output
```
(a^2*sqrt(abs(a)) + a*abs(a)^(3/2))*arctan(1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a
)) + 2*sqrt(a*tan(1/2*x)))/sqrt(abs(a))) + (a^2*sqrt(abs(a)) + a*abs(a)^(3
/2))*arctan(-1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) - 2*sqrt(a*tan(1/2*x)))/sqr
t(abs(a))) + 1/2*(a^2*sqrt(abs(a)) - a*abs(a)^(3/2))*log(a*tan(1/2*x) + sq
rt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(a)) - 1/2*(a^2*sqrt(abs(a)) -
a*abs(a)^(3/2))*log(a*tan(1/2*x) - sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a))
 + abs(a)) + 1/6*sqrt(2)*(sqrt(a*tan(1/2*x))*a^2*tan(1/2*x) + 15*sqrt(a*ta
n(1/2*x))*a^2) - 1/6*sqrt(2)*(15*a^4*tan(1/2*x) + a^4)/(sqrt(a*tan(1/2*x))
*a*tan(1/2*x))
```

3.13.9 Mupad [F(-1)]

Timed out.

$$\int (a + a\csc(x))^{5/2}\, dx = \int \left(a + \frac{a}{\sin(x)}\right)^{5/2} dx$$

input
```
int((a + a/sin(x))^(5/2),x)
```

output
```
int((a + a/sin(x))^(5/2), x)
```

3.14 $\int (a+a\csc(x))^{3/2}\,dx$

3.14.1 Optimal result

Integrand size = 10, antiderivative size = 44

$$\int (a+a\csc(x))^{3/2}\,dx=-2a^{3/2}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right)-\frac{2a^2\cot(x)}{\sqrt{a+a\csc(x)}}$$

output
```
-2*a^(3/2)*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))-2*a^2*cot(x)/(a+a*csc
(x))^(1/2)
```

3.14.2 Mathematica [A] (verified)

Time = 0.13 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.57

$$\int (a+a\csc(x))^{3/2}\,dx = -\frac{2a\left(\arctan\left(\sqrt{-1+\csc(x)}\right)+\sqrt{-1+\csc(x)}\right)\sqrt{a(1+\csc(x))}\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)}{\sqrt{-1+\csc(x)}\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)}$$

input
```
Integrate[(a + a*Csc[x])^(3/2),x]
```

output
```
(-2*a*(ArcTan[Sqrt[-1 + Csc[x]]] + Sqrt[-1 + Csc[x]])*Sqrt[a*(1 + Csc[x])]
*(Cos[x/2] - Sin[x/2]))/(Sqrt[-1 + Csc[x]]*(Cos[x/2] + Sin[x/2]))
```

3.14.3 Rubi [A] (verified)

Time = 0.26 (sec) , antiderivative size = 44, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$, Rules used = {3042, 4262, 27, 3042, 4261, 216}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (a\csc(x)+a)^{3/2}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int (a\csc(x)+a)^{3/2}dx$$

$$\Big\downarrow \text{4262}$$

$$2a\int \frac{1}{2}\sqrt{\csc(x)a+a}dx-\frac{2a^2\cot(x)}{\sqrt{a\csc(x)+a}}$$

$$\Big\downarrow \text{27}$$

$$a\int \sqrt{\csc(x)a+a}dx-\frac{2a^2\cot(x)}{\sqrt{a\csc(x)+a}}$$

$$\Big\downarrow \text{3042}$$

$$a\int \sqrt{\csc(x)a+a}dx-\frac{2a^2\cot(x)}{\sqrt{a\csc(x)+a}}$$

$$\Big\downarrow \text{4261}$$

$$-2a^2\int \frac{1}{\frac{a^2\cot^2(x)}{\csc(x)a+a}+a}d\frac{a\cot(x)}{\sqrt{\csc(x)a+a}}-\frac{2a^2\cot(x)}{\sqrt{a\csc(x)+a}}$$

$$\Big\downarrow \text{216}$$

$$-2a^{3/2}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)-\frac{2a^2\cot(x)}{\sqrt{a\csc(x)+a}}$$

input
```
Int[(a + a*Csc[x])^(3/2),x]
```

output
```
-2*a^(3/2)*ArcTan[(Sqrt[a]*Cot[x])/Sqrt[a + a*Csc[x]]] - (2*a^2*Cot[x])/Sq
rt[a + a*Csc[x]]
```

3.14.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 216
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[b, 2]))*A
rcTan[Rt[b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4261
```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Simp[-2*(b/d)
  Subst[Int[1/(a + x^2), x], x, b*(Cot[c + d*x]/Sqrt[a + b*Csc[c + d*x]])],
 x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

rule 4262
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(-b^2)*C
ot[c + d*x]*((a + b*Csc[c + d*x])^(n - 2)/(d*(n - 1))), x] + Simp[a/(n - 1)
   Int[(a + b*Csc[c + d*x])^(n - 2)*(a*(n - 1) + b*(3*n - 4)*Csc[c + d*x]),
 x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && GtQ[n, 1] && Inte
gerQ[2*n]
```

3.14.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 244 vs. $2(36) = 72$.

Time = 0.55 (sec) , antiderivative size = 245, normalized size of antiderivative = 5.57

method	result
default	$\frac{\csc(x)\left(\frac{a\left(\csc(x)(1-\cos(x))^2+2-2\cos(x)+\sin(x)\right)}{1-\cos(x)}\right)^{\frac{3}{2}}(1-\cos(x))\left(\sqrt{\csc(x)-\cot(x)}\,\sqrt{2}\,\ln\left(-\frac{\csc(x)-\cot(x)+\sqrt{\csc(x)-\cot(x)}\,\sqrt{2}+1}{\sqrt{\csc(x)-\cot(x)}\,\sqrt{2}-\csc(x)+\cot(x)-1}\right)+4\sqrt{2}}{}$

input
```
int((a+a*csc(x))^(3/2),x,method=_RETURNVERBOSE)
```

output
```
1/4*csc(x)*(a/(1-cos(x))*(csc(x)*(1-cos(x))^2+2-2*cos(x)+sin(x)))^(3/2)/(c
sc(x)-cot(x)+1)^3*(1-cos(x))*((csc(x)-cot(x))^(1/2)*2^(1/2)*ln(-(csc(x)-co
t(x)+(csc(x)-cot(x))^(1/2)*2^(1/2)+1)/((csc(x)-cot(x))^(1/2)*2^(1/2)-csc(x
)+cot(x)-1))+4*2^(1/2)*(csc(x)-cot(x))^(1/2)*arctan((csc(x)-cot(x))^(1/2)*
2^(1/2)+1)+4*2^(1/2)*(csc(x)-cot(x))^(1/2)*arctan((csc(x)-cot(x))^(1/2)*2^
(1/2)-1)+(csc(x)-cot(x))^(1/2)*2^(1/2)*ln(-((csc(x)-cot(x))^(1/2)*2^(1/2)-
csc(x)+cot(x)-1)/(csc(x)-cot(x)+(csc(x)-cot(x))^(1/2)*2^(1/2)+1))+4*csc(x)
-4*cot(x)-4)*2^(1/2)
```

3.14.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 90 vs. $2(36) = 72$.

Time = 0.26 (sec) , antiderivative size = 212, normalized size of antiderivative = 4.82

$$\int (a + a\csc(x))^{3/2}\, dx = \left[\frac{(a\cos(x) + a\sin(x) + a)\sqrt{-a}\log\left(\frac{2\,a\cos(x)^2 - 2\left(\cos(x)^2 + (\cos(x)+1)\sin(x) - 1\right)\sqrt{-a}\sqrt{\frac{a\sin(x)+a}{\sin(x)}} + a\cos}{\cos(x)+\sin(x)+1}\right.}{\cos(x) + \sin(x) + 1}\right.$$

input
```
integrate((a+a*csc(x))^(3/2),x, algorithm="fricas")
```

output
```
[((a*cos(x) + a*sin(x) + a)*sqrt(-a)*log((2*a*cos(x)^2 - 2*(cos(x)^2 + (co
s(x) + 1)*sin(x) - 1)*sqrt(-a)*sqrt((a*sin(x) + a)/sin(x)) + a*cos(x) - (2
*a*cos(x) + a)*sin(x) - a)/(cos(x) + sin(x) + 1)) - 2*(a*cos(x) - a*sin(x)
 + a)*sqrt((a*sin(x) + a)/sin(x)))/(cos(x) + sin(x) + 1), 2*((a*cos(x) + a
*sin(x) + a)*sqrt(a)*arctan(-sqrt(a)*sqrt((a*sin(x) + a)/sin(x))*(cos(x) -
 sin(x) + 1)/(a*cos(x) + a*sin(x) + a)) - (a*cos(x) - a*sin(x) + a)*sqrt((
a*sin(x) + a)/sin(x)))/(cos(x) + sin(x) + 1)]
```

3.14.6 Sympy [F]

$$\int (a+a\csc(x))^{3/2}\,dx=\int \left(a\csc{(x)}+a\right)^{\frac{3}{2}}\,dx$$

input
```
integrate((a+a*csc(x))**(3/2),x)
```

output
```
Integral((a*csc(x) + a)**(3/2), x)
```

3.14.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 200 vs. 2(36) = 72.

Time = 0.33 (sec) , antiderivative size = 200, normalized size of antiderivative = 4.55

$$\int (a+a\csc(x))^{3/2}\,dx=\sqrt{2}\left(\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\sqrt{2}+2\sqrt{\frac{\sin{(x)}}{\cos{(x)}+1}}\right)\right)+\sqrt{2}\arctan\left(-\frac{1}{2}\sqrt{2}\left(\sqrt{2}-2\sqrt{\frac{\sin{(x)}}{\cos{(x)}+1}}\right)\right)\right)a^{\frac{3}{2}} - \frac{1}{5}\sqrt{2}\left(a^{\frac{3}{2}}\left(\frac{\sin{(x)}}{\cos{(x)}+1}\right)^{\frac{5}{2}}+5\,a^{\frac{3}{2}}\left(\frac{\sin{(x)}}{\cos{(x)}+1}\right)^{\frac{3}{2}}+10\,a^{\frac{3}{2}}\sqrt{\frac{\sin{(x)}}{\cos{(x)}+1}}\right) - \frac{\frac{5\sqrt{2}a^{\frac{3}{2}}\sin(x)}{\cos(x)+1}-\frac{15\sqrt{2}a^{\frac{3}{2}}\sin(x)^2}{(\cos(x)+1)^2}-\frac{5\sqrt{2}a^{\frac{3}{2}}\sin(x)^3}{(\cos(x)+1)^3}-\frac{\sqrt{2}a^{\frac{3}{2}}\sin(x)^4}{(\cos(x)+1)^4}}{5\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{3}{2}}}$$

input
```
integrate((a+a*csc(x))^(3/2),x, algorithm="maxima")
```

output
```
sqrt(2)*(sqrt(2)*arctan(1/2*sqrt(2)*(sqrt(2) + 2*sqrt(sin(x)/(cos(x) + 1))
)) + sqrt(2)*arctan(-1/2*sqrt(2)*(sqrt(2) - 2*sqrt(sin(x)/(cos(x) + 1)))))
*a^(3/2) - 1/5*sqrt(2)*(a^(3/2)*(sin(x)/(cos(x) + 1))^(5/2) + 5*a^(3/2)*(s
in(x)/(cos(x) + 1))^(3/2) + 10*a^(3/2)*sqrt(sin(x)/(cos(x) + 1))) - 1/5*(5
*sqrt(2)*a^(3/2)*sin(x)/(cos(x) + 1) - 15*sqrt(2)*a^(3/2)*sin(x)^2/(cos(x)
 + 1)^2 - 5*sqrt(2)*a^(3/2)*sin(x)^3/(cos(x) + 1)^3 - sqrt(2)*a^(3/2)*sin(
x)^4/(cos(x) + 1)^4)/(sin(x)/(cos(x) + 1))^(3/2)
```

3.14.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 195 vs. $2(36) = 72$.

Time = 0.42 (sec) , antiderivative size = 195, normalized size of antiderivative = 4.43

$$\int (a + a\csc(x))^{3/2}\,dx = \sqrt{2}\sqrt{a\tan\left(\frac{1}{2}\,x\right)}a - \frac{\sqrt{2}a^2}{\sqrt{a\tan\left(\frac{1}{2}\,x\right)}} + \left(a\sqrt{|a|} + |a|^{\frac{3}{2}}\right)\arctan\left(\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|} + 2\sqrt{a\tan\left(\frac{1}{2}\,x\right)}\right)}{2\sqrt{|a|}}\right) + \left(a\sqrt{|a|} + |a|^{\frac{3}{2}}\right)\arctan\left(-\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|} - 2\sqrt{a\tan\left(\frac{1}{2}\,x\right)}\right)}{2\sqrt{|a|}}\right) + \frac{1}{2}\left(a\sqrt{|a|} - |a|^{\frac{3}{2}}\right)\log\left(a\tan\left(\frac{1}{2}\,x\right) + \sqrt{2}\sqrt{a\tan\left(\frac{1}{2}\,x\right)}\sqrt{|a|} + |a|\right) - \frac{1}{2}\left(a\sqrt{|a|} - |a|^{\frac{3}{2}}\right)\log\left(a\tan\left(\frac{1}{2}\,x\right) - \sqrt{2}\sqrt{a\tan\left(\frac{1}{2}\,x\right)}\sqrt{|a|} + |a|\right)$$

input
```
integrate((a+a*csc(x))^(3/2),x, algorithm="giac")
```

output
```
sqrt(2)*sqrt(a*tan(1/2*x))*a - sqrt(2)*a^2/sqrt(a*tan(1/2*x)) + (a*sqrt(ab
s(a)) + abs(a)^(3/2))*arctan(1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) + 2*sqrt(a*
tan(1/2*x)))/sqrt(abs(a))) + (a*sqrt(abs(a)) + abs(a)^(3/2))*arctan(-1/2*s
qrt(2)*(sqrt(2)*sqrt(abs(a)) - 2*sqrt(a*tan(1/2*x)))/sqrt(abs(a))) + 1/2*(
a*sqrt(abs(a)) - abs(a)^(3/2))*log(a*tan(1/2*x) + sqrt(2)*sqrt(a*tan(1/2*x
))*sqrt(abs(a)) + abs(a)) - 1/2*(a*sqrt(abs(a)) - abs(a)^(3/2))*log(a*tan(
1/2*x) - sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(a))
```

3.14.9 Mupad [F(-1)]

Timed out.

$$\int (a + a\csc(x))^{3/2}\,dx = \int \left(a + \frac{a}{\sin(x)}\right)^{3/2}\,dx$$

input
```
int((a + a/sin(x))^(3/2),x)
```

output
```
int((a + a/sin(x))^(3/2), x)
```

3.14. $\int (a + a\csc(x))^{3/2}\, dx$

3.15 $\int \sqrt{a+a\csc(x)}\,dx$

3.15.1 Optimal result

Integrand size = 10, antiderivative size = 26

$$\int \sqrt{a+a\csc(x)}\,dx=-2\sqrt{a}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right)$$

output
```
-2*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))*a^(1/2)
```

3.15.2 Mathematica [A] (verified)

Time = 0.08 (sec) , antiderivative size = 32, normalized size of antiderivative = 1.23

$$\int \sqrt{a+a\csc(x)}\,dx=-\frac{2a\arctan\left(\sqrt{-1+\csc(x)}\right)\cot(x)}{\sqrt{-1+\csc(x)}\sqrt{a(1+\csc(x))}}$$

input
```
Integrate[Sqrt[a + a*Csc[x]],x]
```

output
```
(-2*a*ArcTan[Sqrt[-1 + Csc[x]]]*Cot[x])/(Sqrt[-1 + Csc[x]]*Sqrt[a*(1 + Csc
[x])])
```

3.15.3 Rubi [A] (verified)

Time = 0.17 (sec) , antiderivative size = 26, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.300, Rules used = {3042, 4261, 216}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sqrt{a\csc(x)+a}\,dx$$
$$\Big\downarrow \text{3042}$$
$$\int \sqrt{a\csc(x)+a}dx$$
$$\Big\downarrow \text{4261}$$
$$-2a\int \frac{1}{\frac{a^2\cot^2(x)}{\csc(x)a+a}+a}d\frac{a\cot(x)}{\sqrt{\csc(x)a+a}}$$
$$\Big\downarrow \text{216}$$
$$-2\sqrt{a}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)$$

input
```
Int[Sqrt[a + a*Csc[x]],x]
```

output
```
-2*Sqrt[a]*ArcTan[(Sqrt[a]*Cot[x])/Sqrt[a + a*Csc[x]]]
```

3.15.3.1 Defintions of rubi rules used

rule 216
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[b, 2]))*A
rcTan[Rt[b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4261
```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Simp[-2*(b/d)
  Subst[Int[1/(a + x^2), x], x, b*(Cot[c + d*x]/Sqrt[a + b*Csc[c + d*x]])],
 x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

3.15.4 Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 165 vs. $2(20) = 40$.

Time = 1.06 (sec) , antiderivative size = 166, normalized size of antiderivative = 6.38

method	result
default	$\frac{\sqrt{2}\,\sqrt{a(\csc(x)+1)}\,\sin(x)\sqrt{\csc(x)-\cot(x)}\,\left(\ln\left(\frac{\csc(x)-\cot(x)+\sqrt{\csc(x)-\cot(x)}\,\sqrt{2}+1}{-\sqrt{\csc(x)-\cot(x)}\,\sqrt{2}+\csc(x)-\cot(x)+1}\right)+4\arctan\left(\sqrt{\csc(x)-\cot(x)}\,\sqrt{2}+1\right)+4\arctan\left(\right.\right.}{2-2\cos(x)+2\sin(x)}$

input
```
int((a+a*csc(x))^(1/2),x,method=_RETURNVERBOSE)
```

output
```
1/2*2^(1/2)*(a*(csc(x)+1))^(1/2)*sin(x)*(csc(x)-cot(x))^(1/2)*(ln((csc(x)-
cot(x)+(csc(x)-cot(x))^(1/2)*2^(1/2)+1)/(-(csc(x)-cot(x))^(1/2)*2^(1/2)+cs
c(x)-cot(x)+1))+4*arctan((csc(x)-cot(x))^(1/2)*2^(1/2)+1)+4*arctan((csc(x)
-cot(x))^(1/2)*2^(1/2)-1)+ln((-(csc(x)-cot(x))^(1/2)*2^(1/2)+csc(x)-cot(x)
+1)/(csc(x)-cot(x)+(csc(x)-cot(x))^(1/2)*2^(1/2)+1)))/(1-cos(x)+sin(x))
```

3.15.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 44 vs. $2(20) = 40$.

Time = 0.26 (sec) , antiderivative size = 120, normalized size of antiderivative = 4.62

$$\int \sqrt{a + a\csc(x)}\,dx$$
$$= \left[\sqrt{-a}\log\left(\frac{2\,a\cos(x)^2 - 2\left(\cos(x)^2 + (\cos(x)+1)\sin(x) - 1\right)\sqrt{-a}\sqrt{\frac{a\sin(x)+a}{\sin(x)}} + a\cos(x) - (2\,a\cos(x)\cdots}{\cos(x) + \sin(x) + 1}\right.\right.$$

input
```
integrate((a+a*csc(x))^(1/2),x, algorithm="fricas")
```

output
```
[sqrt(-a)*log((2*a*cos(x)^2 - 2*(cos(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt(
-a)*sqrt((a*sin(x) + a)/sin(x)) + a*cos(x) - (2*a*cos(x) + a)*sin(x) - a)/
(cos(x) + sin(x) + 1)), 2*sqrt(a)*arctan(-sqrt(a)*sqrt((a*sin(x) + a)/sin(
x))*(cos(x) - sin(x) + 1)/(a*cos(x) + a*sin(x) + a))]
```

3.15.6 Sympy [F]

$$\int \sqrt{a + a\csc(x)}\,dx = \int \sqrt{a\csc{(x)} + a}\,dx$$

input
```
integrate((a+a*csc(x))**(1/2),x)
```

output
```
Integral(sqrt(a*csc(x) + a), x)
```

3.15.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 148 vs. $2(20) = 40$.

Time = 0.32 (sec) , antiderivative size = 148, normalized size of antiderivative = 5.69

$$\int \sqrt{a + a\csc(x)}\,dx = -\frac{2}{3}\sqrt{2}\sqrt{a}\left(\frac{\sin{(x)}}{\cos{(x)}+1}\right)^{\frac{3}{2}} + \sqrt{2}\left(\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\sqrt{2}+2\sqrt{\frac{\sin{(x)}}{\cos{(x)}+1}}\right)\right) + \sqrt{2}\arctan\left(-\frac{1}{2}\sqrt{2}\left(\sqrt{2}-2\sqrt{\frac{\sin{(x)}}{\cos{(x)}+1}}\right)\right)\right)\sqrt{} - 2\sqrt{2}\sqrt{a}\sqrt{\frac{\sin{(x)}}{\cos{(x)}+1}} + \frac{2\left(\frac{3\sqrt{2}\sqrt{a}\sin(x)}{\cos(x)+1} + \frac{\sqrt{2}\sqrt{a}\sin(x)^2}{(\cos(x)+1)^2}\right)}{3\sqrt{\frac{\sin(x)}{\cos(x)+1}}}$$

input
```
integrate((a+a*csc(x))^(1/2),x, algorithm="maxima")
```

output
```
-2/3*sqrt(2)*sqrt(a)*(sin(x)/(cos(x) + 1))^(3/2) + sqrt(2)*(sqrt(2)*arctan
(1/2*sqrt(2)*(sqrt(2) + 2*sqrt(sin(x)/(cos(x) + 1)))) + sqrt(2)*arctan(-1/
2*sqrt(2)*(sqrt(2) - 2*sqrt(sin(x)/(cos(x) + 1)))))*sqrt(a) - 2*sqrt(2)*sq
rt(a)*sqrt(sin(x)/(cos(x) + 1)) + 2/3*(3*sqrt(2)*sqrt(a)*sin(x)/(cos(x) +
1) + sqrt(2)*sqrt(a)*sin(x)^2/(cos(x) + 1)^2)/sqrt(sin(x)/(cos(x) + 1))
```

3.15.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 353 vs. 2(20) = 40.

Time = 0.48 (sec) , antiderivative size = 353, normalized size of antiderivative = 13.58

$$\int \sqrt{a + a\csc(x)}\, dx$$
$$= \frac{1}{4}\sqrt{2}\left(\frac{2\sqrt{2}\left(a\sqrt{|a|}\operatorname{sgn}\left(\tan\left(\frac{1}{2}x\right)^3 + \tan\left(\frac{1}{2}x\right)^2 + \tan\left(\frac{1}{2}x\right) + 1\right) + |a|^{\frac{3}{2}}\operatorname{sgn}\left(\tan\left(\frac{1}{2}x\right)^3 + \tan\left(\frac{1}{2}x\right)^2 + \mathrm{ta}\right.}{a}\right.$$

input
```
integrate((a+a*csc(x))^(1/2),x, algorithm="giac")
```

output
```
1/4*sqrt(2)*(2*sqrt(2)*(a*sqrt(abs(a))*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + t
an(1/2*x) + 1) + abs(a)^(3/2)*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + tan(1/2*x)
 + 1))*arctan(1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) + 2*sqrt(a*tan(1/2*x)))/sq
rt(abs(a)))/a + 2*sqrt(2)*(a*sqrt(abs(a))*sgn(tan(1/2*x)^3 + tan(1/2*x)^2
+ tan(1/2*x) + 1) + abs(a)^(3/2)*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + tan(1/2
*x) + 1))*arctan(-1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) - 2*sqrt(a*tan(1/2*x))
)/sqrt(abs(a)))/a + sqrt(2)*(a*sqrt(abs(a))*sgn(tan(1/2*x)^3 + tan(1/2*x)^
2 + tan(1/2*x) + 1) - abs(a)^(3/2)*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + tan(1
/2*x) + 1))*log(a*tan(1/2*x) + sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + a
bs(a))/a - sqrt(2)*(a*sqrt(abs(a))*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + tan(1
/2*x) + 1) - abs(a)^(3/2)*sgn(tan(1/2*x)^3 + tan(1/2*x)^2 + tan(1/2*x) + 1
))*log(a*tan(1/2*x) - sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(a))/a)
*sgn(sin(x))
```

3.15.9 Mupad [F(-1)]

Timed out.

$$\int \sqrt{a + a\csc(x)}\, dx = \int \sqrt{a + \frac{a}{\sin(x)}}\, dx$$

input
```
int((a + a/sin(x))^(1/2),x)
```

output
```
int((a + a/sin(x))^(1/2), x)
```

3.16 $\int \frac{1}{\sqrt{a+a\csc(x)}}\,dx$

3.16.1 Optimal result

Integrand size = 10, antiderivative size = 62

$$\int \frac{1}{\sqrt{a+a\csc(x)}}\,dx=-\frac{2\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right)}{\sqrt{a}}+\frac{\sqrt{2}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a+a\csc(x)}}\right)}{\sqrt{a}}$$

output
```
-2*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))/a^(1/2)+arctan(1/2*cot(x)*a^(
1/2)*2^(1/2)/(a+a*csc(x))^(1/2))*2^(1/2)/a^(1/2)
```

3.16.2 Mathematica [A] (verified)

Time = 0.18 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.87

$$\int \frac{1}{\sqrt{a+a\csc(x)}}\,dx=\frac{\left(-2\arctan\left(\sqrt{-1+\csc(x)}\right)+\sqrt{2}\arctan\left(\frac{\sqrt{-1+\csc(x)}}{\sqrt{2}}\right)\right)\cot(x)}{\sqrt{-1+\csc(x)}\sqrt{a(1+\csc(x))}}$$

input
```
Integrate[1/Sqrt[a + a*Csc[x]],x]
```

output
```
((-2*ArcTan[Sqrt[-1 + Csc[x]]] + Sqrt[2]*ArcTan[Sqrt[-1 + Csc[x]]/Sqrt[2]]
)*Cot[x])/(Sqrt[-1 + Csc[x]]*Sqrt[a*(1 + Csc[x])])
```

3.16.3 Rubi [A] (verified)

Time = 0.33 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.700, Rules used = {3042, 4263, 3042, 4261, 216, 4282, 216}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{\sqrt{a \csc(x)+a}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\sqrt{a \csc(x)+a}}dx$$

$$\Big\downarrow \text{4263}$$

$$\frac{\int \sqrt{\csc(x)a+a}dx}{a}-\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \sqrt{\csc(x)a+a}dx}{a}-\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx$$

$$\Big\downarrow \text{4261}$$

$$-2\int \frac{1}{\frac{a^2\cot^2(x)}{\csc(x)a+a}+a}d\frac{a\cot(x)}{\sqrt{\csc(x)a+a}}-\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx$$

$$\Big\downarrow \text{216}$$

$$-\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx-\frac{2\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{\sqrt{a}}$$

$$\Big\downarrow \text{4282}$$

$$2\int \frac{1}{\frac{a^2\cot^2(x)}{\csc(x)a+a}+2a}d\frac{a\cot(x)}{\sqrt{\csc(x)a+a}}-\frac{2\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{\sqrt{a}}$$

$$\Big\downarrow \text{216}$$

$$\frac{\sqrt{2}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a\csc(x)+a}}\right)}{\sqrt{a}}-\frac{2\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{\sqrt{a}}$$

input
```
Int[1/Sqrt[a + a*Csc[x]],x]
```

output
```
(-2*ArcTan[(Sqrt[a]*Cot[x])/Sqrt[a + a*Csc[x]]])/Sqrt[a] + (Sqrt[2]*ArcTan
[(Sqrt[a]*Cot[x])/(Sqrt[2]*Sqrt[a + a*Csc[x]])])/Sqrt[a]
```

3.16.3.1 Defintions of rubi rules used

rule 216
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[b, 2]))*A
rcTan[Rt[b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4261
```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Simp[-2*(b/d)
  Subst[Int[1/(a + x^2), x], x, b*(Cot[c + d*x]/Sqrt[a + b*Csc[c + d*x]])],
 x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

rule 4263
```
Int[1/Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Simp[1/a   I
nt[Sqrt[a + b*Csc[c + d*x]], x], x] - Simp[b/a   Int[Csc[c + d*x]/Sqrt[a +
b*Csc[c + d*x]], x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

rule 4282
```
Int[csc[(e_.) + (f_.)*(x_)]/Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)], x_S
ymbol] :> Simp[-2/f   Subst[Int[1/(2*a + x^2), x], x, b*(Cot[e + f*x]/Sqrt[
a + b*Csc[e + f*x]])], x] /; FreeQ[{a, b, e, f}, x] && EqQ[a^2 - b^2, 0]
```

3.16.4 Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 184 vs. $2(47) = 94$.

Time = 0.53 (sec) , antiderivative size = 185, normalized size of antiderivative = 2.98

method	result
default	$\frac{\left(\sqrt{2}\,\ln\left(\frac{\csc(x)-\cot(x)+\sqrt{\csc(x)-\cot(x)}\,\sqrt{2}+1}{-\sqrt{\csc(x)-\cot(x)}\,\sqrt{2}+\csc(x)-\cot(x)+1}\right)+4\sqrt{2}\,\arctan\left(\sqrt{\csc(x)-\cot(x)}\,\sqrt{2}+1\right)+4\sqrt{2}\,\arctan\left(\sqrt{\csc(x)-\cot(x)}\,\sqrt{2}-1\right)+\sqrt{2}}{4\sqrt{a(\csc(x)+1)}\,\sqrt{\csc(x)-\cot(x)}}$

input
```
int(1/(a+a*csc(x))^(1/2),x,method=_RETURNVERBOSE)
```

output
```
1/4*(2^(1/2)*ln((csc(x)-cot(x)+(csc(x)-cot(x))^(1/2)*2^(1/2)+1)/(-(csc(x)-
cot(x))^(1/2)*2^(1/2)+csc(x)-cot(x)+1))+4*2^(1/2)*arctan((csc(x)-cot(x))^(
1/2)*2^(1/2)+1)+4*2^(1/2)*arctan((csc(x)-cot(x))^(1/2)*2^(1/2)-1)+2^(1/2)*
ln((-(csc(x)-cot(x))^(1/2)*2^(1/2)+csc(x)-cot(x)+1)/(csc(x)-cot(x)+(csc(x)
-cot(x))^(1/2)*2^(1/2)+1))-8*arctan((csc(x)-cot(x))^(1/2)))/(a*(csc(x)+1))
^(1/2)/(csc(x)-cot(x))^(1/2)*(csc(x)-cot(x)+1)
```

3.16.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 219, normalized size of antiderivative = 3.53

$$\int \frac{1}{\sqrt{a+a\csc(x)}}\,dx$$
$$= \left[\frac{\sqrt{2}a\sqrt{-\frac{1}{a}}\log\left(\frac{\sqrt{2}\sqrt{\frac{a\sin(x)+a}{\sin(x)}}\sqrt{-\frac{1}{a}}\sin(x)+\cos(x)}{\sin(x)+1}\right)-\sqrt{-a}\log\left(\frac{2a\cos(x)^2+2\left(\cos(x)^2+(\cos(x)+1)\sin(x)-1\right)\sqrt{-a}\sqrt{\frac{a\sin(x)+}{\sin(x)}}}{\cos(x)+\sin(x)+1}\right.}{a}\right.$$
$$\left.-\frac{2\left(\sqrt{2}\sqrt{a}\arctan\left(\frac{\sqrt{2}\sqrt{\frac{a\sin(x)+a}{\sin(x)}}\sin(x)}{\sqrt{a}(\cos(x)+\sin(x)+1)}\right)-\sqrt{a}\arctan\left(-\frac{\sqrt{a}\sqrt{\frac{a\sin(x)+a}{\sin(x)}}(\cos(x)-\sin(x)+1)}{a\cos(x)+a\sin(x)+a}\right)\right)}{a}\right]$$

input
```
integrate(1/(a+a*csc(x))^(1/2),x, algorithm="fricas")
```

output
```
[(sqrt(2)*a*sqrt(-1/a)*log((sqrt(2)*sqrt((a*sin(x) + a)/sin(x))*sqrt(-1/a)
*sin(x) + cos(x))/(sin(x) + 1)) - sqrt(-a)*log((2*a*cos(x)^2 + 2*(cos(x)^2
 + (cos(x) + 1)*sin(x) - 1)*sqrt(-a)*sqrt((a*sin(x) + a)/sin(x)) + a*cos(x
) - (2*a*cos(x) + a)*sin(x) - a)/(cos(x) + sin(x) + 1)))/a, -2*(sqrt(2)*sq
rt(a)*arctan(sqrt(2)*sqrt((a*sin(x) + a)/sin(x))*sin(x)/(sqrt(a)*(cos(x) +
 sin(x) + 1))) - sqrt(a)*arctan(-sqrt(a)*sqrt((a*sin(x) + a)/sin(x))*(cos(
x) - sin(x) + 1)/(a*cos(x) + a*sin(x) + a)))/a]
```

3.16.6 Sympy [F]

$$\int \frac{1}{\sqrt{a+a\csc(x)}}\,dx=\int \frac{1}{\sqrt{a\csc{(x)}+a}}\,dx$$

input
```
integrate(1/(a+a*csc(x))**(1/2),x)
```

output
```
Integral(1/sqrt(a*csc(x) + a), x)
```

3.16.7 Maxima [A] (verification not implemented)

Time = 0.33 (sec) , antiderivative size = 83, normalized size of antiderivative = 1.34

$$\int \frac{1}{\sqrt{a+a\csc(x)}}\,dx = \frac{\sqrt{2}\left(\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\sqrt{2}+2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right)+\sqrt{2}\arctan\left(-\frac{1}{2}\sqrt{2}\left(\sqrt{2}-2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right)\right)}{\sqrt{a}} - \frac{2\sqrt{2}\arctan\left(\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)}{\sqrt{a}}$$

input
```
integrate(1/(a+a*csc(x))^(1/2),x, algorithm="maxima")
```

output
```
sqrt(2)*(sqrt(2)*arctan(1/2*sqrt(2)*(sqrt(2) + 2*sqrt(sin(x)/(cos(x) + 1))
)) + sqrt(2)*arctan(-1/2*sqrt(2)*(sqrt(2) - 2*sqrt(sin(x)/(cos(x) + 1)))))
/sqrt(a) - 2*sqrt(2)*arctan(sqrt(sin(x)/(cos(x) + 1)))/sqrt(a)
```

3.16.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 205 vs. $2(47) = 94$.

Time = 0.43 (sec) , antiderivative size = 205, normalized size of antiderivative = 3.31

$$\int \frac{1}{\sqrt{a+a\csc(x)}}\,dx = -\frac{4\sqrt{2}\sqrt{a}\arctan\left(\frac{\sqrt{a\tan\left(\frac{1}{2}x\right)}}{\sqrt{a}}\right) - \frac{2\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)\arctan\left(\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|}+2\sqrt{a\tan\left(\frac{1}{2}x\right)}\right)}{2\sqrt{|a|}}\right)}{a} - \frac{2\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)\arctan\left(-\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|\cdots}\right.}{\cdots}\right.}{a}\cdots}{\cdots}$$

input
```
integrate(1/(a+a*csc(x))^(1/2),x, algorithm="giac")
```

output
```
-1/2*(4*sqrt(2)*sqrt(a)*arctan(sqrt(a*tan(1/2*x))/sqrt(a)) - 2*(a*sqrt(abs
(a)) + abs(a)^(3/2))*arctan(1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) + 2*sqrt(a*t
an(1/2*x)))/sqrt(abs(a)))/a - 2*(a*sqrt(abs(a)) + abs(a)^(3/2))*arctan(-1/
2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) - 2*sqrt(a*tan(1/2*x)))/sqrt(abs(a)))/a -
(a*sqrt(abs(a)) - abs(a)^(3/2))*log(a*tan(1/2*x) + sqrt(2)*sqrt(a*tan(1/2*
x))*sqrt(abs(a)) + abs(a))/a + (a*sqrt(abs(a)) - abs(a)^(3/2))*log(a*tan(1
/2*x) - sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(a))/a)/a
```

3.16.9 Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{a + a\csc(x)}}\, dx = \int \frac{1}{\sqrt{a + \frac{a}{\sin(x)}}}\, dx$$

input
```
int(1/(a + a/sin(x))^(1/2),x)
```

output
```
int(1/(a + a/sin(x))^(1/2), x)
```

3.17 $\int \frac{1}{(a+a\csc(x))^{3/2}}\,dx$

3.17.1 Optimal result

Integrand size = 10, antiderivative size = 81

$$\int \frac{1}{(a+a\csc(x))^{3/2}}\,dx = -\frac{2\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right)}{a^{3/2}} + \frac{5\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a+a\csc(x)}}\right)}{2\sqrt{2}a^{3/2}} + \frac{\cot(x)}{2(a+a\csc(x))^{3/2}}$$

output
```
-2*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))/a^(3/2)+1/2*cot(x)/(a+a*csc(x
))^(3/2)+5/4*arctan(1/2*cot(x)*a^(1/2)*2^(1/2)/(a+a*csc(x))^(1/2))/a^(3/2)
*2^(1/2)
```

3.17.2 Mathematica [A] (verified)

Time = 0.50 (sec) , antiderivative size = 129, normalized size of antiderivative = 1.59

$$\int \frac{1}{(a+a\csc(x))^{3/2}}\,dx = -\frac{\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)\left(2-2\csc(x)+8\arctan\left(\sqrt{-1+\csc(x)}\right)\sqrt{-1+\csc(x)}(1+\csc(x))-5\sqrt{2}\arctan\left(\right.}{4a(-1+\csc(x))\sqrt{a(1+\csc(x))}\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)}$$

input
```
Integrate[(a + a*Csc[x])^(-3/2),x]
```

output
```
-1/4*((Cos[x/2] - Sin[x/2])*(2 - 2*Csc[x] + 8*ArcTan[Sqrt[-1 + Csc[x]]]*Sq
rt[-1 + Csc[x]]*(1 + Csc[x]) - 5*Sqrt[2]*ArcTan[Sqrt[-1 + Csc[x]]/Sqrt[2]]
*Sqrt[-1 + Csc[x]]*Csc[x]*(Cos[x/2] + Sin[x/2])^2))/(a*(-1 + Csc[x])*Sqrt[
a*(1 + Csc[x])]*(Cos[x/2] + Sin[x/2]))
```

3.17.3 Rubi [A] (verified)

Time = 0.46 (sec) , antiderivative size = 87, normalized size of antiderivative = 1.07, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}} = 1.000$, Rules used = {3042, 4264, 27, 3042, 4408, 3042, 4261, 216, 4282, 216}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{(a\csc(x)+a)^{3/2}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{(a\csc(x)+a)^{3/2}}dx$$

$$\Big\downarrow \text{4264}$$

$$\frac{\cot(x)}{2(a\csc(x)+a)^{3/2}}-\frac{\int -\frac{4a-a\csc(x)}{2\sqrt{\csc(x)a+a}}dx}{2a^2}$$

$$\Big\downarrow \text{27}$$

$$\frac{\int \frac{4a-a\csc(x)}{\sqrt{\csc(x)a+a}}dx}{4a^2}+\frac{\cot(x)}{2(a\csc(x)+a)^{3/2}}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{4a-a\csc(x)}{\sqrt{\csc(x)a+a}}dx}{4a^2}+\frac{\cot(x)}{2(a\csc(x)+a)^{3/2}}$$

$$\Big\downarrow \text{4408}$$

$$\frac{4\int \sqrt{\csc(x)a+a}dx-5a\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx}{4a^2}+\frac{\cot(x)}{2(a\csc(x)+a)^{3/2}}$$

$$\Big\downarrow \text{3042}$$

$$\frac{4\int \sqrt{\csc(x)a+a}dx-5a\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx}{4a^2}+\frac{\cot(x)}{2(a\csc(x)+a)^{3/2}}$$

$$\Big\downarrow \text{4261}$$

$$\frac{-8a\int \frac{1}{\frac{a^2\cot^2(x)}{\csc(x)a+a}+a}d\frac{a\cot(x)}{\sqrt{\csc(x)a+a}}-5a\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx}{4a^2}+\frac{\cot(x)}{2(a\csc(x)+a)^{3/2}}$$

$$\Big\downarrow \text{216}$$

$$\frac{-5a\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx-8\sqrt{a}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{4a^2}+\frac{\cot(x)}{2(a\csc(x)+a)^{3/2}}$$

$$\Big\downarrow \text{4282}$$

$$\frac{10a\int \frac{1}{\frac{a^2\cot^2(x)}{\csc(x)a+a}+2a}d\frac{a\cot(x)}{\sqrt{\csc(x)a+a}}-8\sqrt{a}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{4a^2}+\frac{\cot(x)}{2(a\csc(x)+a)^{3/2}}$$

$$\Big\downarrow \text{216}$$

$$\frac{5\sqrt{2}\sqrt{a}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a\csc(x)+a}}\right)-8\sqrt{a}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{4a^2}+\frac{\cot(x)}{2(a\csc(x)+a)^{3/2}}$$

input
```
Int[(a + a*Csc[x])^(-3/2),x]
```

output
```
(-8*Sqrt[a]*ArcTan[(Sqrt[a]*Cot[x])/Sqrt[a + a*Csc[x]]] + 5*Sqrt[2]*Sqrt[a
]*ArcTan[(Sqrt[a]*Cot[x])/(Sqrt[2]*Sqrt[a + a*Csc[x]])])/(4*a^2) + Cot[x]/
(2*(a + a*Csc[x])^(3/2))
```

3.17.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 216
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[b, 2]))*A
rcTan[Rt[b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4261
```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Simp[-2*(b/d)
  Subst[Int[1/(a + x^2), x], x, b*(Cot[c + d*x]/Sqrt[a + b*Csc[c + d*x]])],
 x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

rule 4264
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(-Cot[c
+ d*x])*((a + b*Csc[c + d*x])^n/(d*(2*n + 1))), x] + Simp[1/(a^2*(2*n + 1))
   Int[(a + b*Csc[c + d*x])^(n + 1)*(a*(2*n + 1) - b*(n + 1)*Csc[c + d*x]),
 x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && LeQ[n, -1] && Int
egerQ[2*n]
```

rule 4282
```
Int[csc[(e_.) + (f_.)*(x_)]/Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)], x_S
ymbol] :> Simp[-2/f   Subst[Int[1/(2*a + x^2), x], x, b*(Cot[e + f*x]/Sqrt[
a + b*Csc[e + f*x]])], x] /; FreeQ[{a, b, e, f}, x] && EqQ[a^2 - b^2, 0]
```

rule 4408
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_
.) + (a_)], x_Symbol] :> Simp[c/a   Int[Sqrt[a + b*Csc[e + f*x]], x], x] -
Simp[(b*c - a*d)/a   Int[Csc[e + f*x]/Sqrt[a + b*Csc[e + f*x]], x], x] /; F
reeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

3.17.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 641 vs. $2(60) = 120$.

Time = 0.54 (sec) , antiderivative size = 642, normalized size of antiderivative = 7.93

method	result	size
default	Expression too large to display	642

input
```
int(1/(a+a*csc(x))^(3/2),x,method=_RETURNVERBOSE)
```

output
```
1/4/(a/(1-cos(x))*(csc(x)*(1-cos(x))^2+2-2*cos(x)+sin(x)))^(3/2)*(csc(x)-c
ot(x)+1)*(csc(x)^2*2^(1/2)*ln(-(csc(x)-cot(x)+(csc(x)-cot(x))^(1/2)*2^(1/2
)+1)/((csc(x)-cot(x))^(1/2)*2^(1/2)-csc(x)+cot(x)-1))*(1-cos(x))^2+4*csc(x
)^2*2^(1/2)*arctan((csc(x)-cot(x))^(1/2)*2^(1/2)+1)*(1-cos(x))^2+4*csc(x)^
2*2^(1/2)*arctan((csc(x)-cot(x))^(1/2)*2^(1/2)-1)*(1-cos(x))^2+csc(x)^2*2^
(1/2)*ln(-((csc(x)-cot(x))^(1/2)*2^(1/2)-csc(x)+cot(x)-1)/(csc(x)-cot(x)+(
csc(x)-cot(x))^(1/2)*2^(1/2)+1))*(1-cos(x))^2+2*2^(1/2)*ln(-(csc(x)-cot(x)
+(csc(x)-cot(x))^(1/2)*2^(1/2)+1)/((csc(x)-cot(x))^(1/2)*2^(1/2)-csc(x)+co
t(x)-1))*(csc(x)-cot(x))+8*2^(1/2)*arctan((csc(x)-cot(x))^(1/2)*2^(1/2)+1)
*(csc(x)-cot(x))+8*2^(1/2)*arctan((csc(x)-cot(x))^(1/2)*2^(1/2)-1)*(csc(x)
-cot(x))+2*2^(1/2)*ln(-((csc(x)-cot(x))^(1/2)*2^(1/2)-csc(x)+cot(x)-1)/(cs
c(x)-cot(x)+(csc(x)-cot(x))^(1/2)*2^(1/2)+1))*(csc(x)-cot(x))-10*csc(x)^2*
arctan((csc(x)-cot(x))^(1/2))*(1-cos(x))^2-2*(csc(x)-cot(x))^(3/2)+2^(1/2)
*ln(-(csc(x)-cot(x)+(csc(x)-cot(x))^(1/2)*2^(1/2)+1)/((csc(x)-cot(x))^(1/2
)*2^(1/2)-csc(x)+cot(x)-1))+4*2^(1/2)*arctan((csc(x)-cot(x))^(1/2)*2^(1/2)
+1)+4*2^(1/2)*arctan((csc(x)-cot(x))^(1/2)*2^(1/2)-1)+2^(1/2)*ln(-((csc(x)
-cot(x))^(1/2)*2^(1/2)-csc(x)+cot(x)-1)/(csc(x)-cot(x)+(csc(x)-cot(x))^(1/
2)*2^(1/2)+1))-20*arctan((csc(x)-cot(x))^(1/2))*(csc(x)-cot(x))-10*arctan(
(csc(x)-cot(x))^(1/2))+2*(csc(x)-cot(x))^(1/2))/(csc(x)-cot(x))^(3/2)*2^(1
/2)
```

3.17.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 195 vs. $2(60) = 120$.

Time = 0.28 (sec) , antiderivative size = 427, normalized size of antiderivative = 5.27

$$\int \frac{1}{(a+a\csc(x))^{3/2}}\,dx = \left[-\frac{5\sqrt{2}\left(\cos(x)^2-(\cos(x)+2)\sin(x)-\cos(x)-2\right)\sqrt{-a}\log\left(-\frac{\sqrt{2}\sqrt{-a}\sqrt{\frac{a\sin(x)+a}{\sin(x)}}}{\sin(x)}\right.}{}\right.$$

input
```
integrate(1/(a+a*csc(x))^(3/2),x, algorithm="fricas")
```

output
```
[-1/4*(5*sqrt(2)*(cos(x)^2 - (cos(x) + 2)*sin(x) - cos(x) - 2)*sqrt(-a)*lo
g(-(sqrt(2)*sqrt(-a)*sqrt((a*sin(x) + a)/sin(x))*sin(x) - a*cos(x))/(sin(x
) + 1)) + 4*(cos(x)^2 - (cos(x) + 2)*sin(x) - cos(x) - 2)*sqrt(-a)*log((2*
a*cos(x)^2 + 2*(cos(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt(-a)*sqrt((a*sin(x
) + a)/sin(x)) + a*cos(x) - (2*a*cos(x) + a)*sin(x) - a)/(cos(x) + sin(x)
+ 1)) + 2*(cos(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt((a*sin(x) + a)/sin(x))
)/(a^2*cos(x)^2 - a^2*cos(x) - 2*a^2 - (a^2*cos(x) + 2*a^2)*sin(x)), 1/2*(
5*sqrt(2)*(cos(x)^2 - (cos(x) + 2)*sin(x) - cos(x) - 2)*sqrt(a)*arctan(sqr
t(2)*sqrt(a)*sqrt((a*sin(x) + a)/sin(x))*(cos(x) + 1)/(a*cos(x) + a*sin(x)
 + a)) + 4*(cos(x)^2 - (cos(x) + 2)*sin(x) - cos(x) - 2)*sqrt(a)*arctan(-s
qrt(a)*sqrt((a*sin(x) + a)/sin(x))*(cos(x) - sin(x) + 1)/(a*cos(x) + a*sin
(x) + a)) - (cos(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt((a*sin(x) + a)/sin(x
)))/(a^2*cos(x)^2 - a^2*cos(x) - 2*a^2 - (a^2*cos(x) + 2*a^2)*sin(x))]
```

3.17.6 Sympy [F]

$$\int \frac{1}{(a+a\csc(x))^{3/2}}\,dx = \int \frac{1}{(a\csc(x)+a)^{\frac{3}{2}}}\,dx$$

input
```
integrate(1/(a+a*csc(x))**(3/2),x)
```

output
```
Integral((a*csc(x) + a)**(-3/2), x)
```

3.17.7 Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 150 vs. $2(60) = 120$.

Time = 0.33 (sec) , antiderivative size = 150, normalized size of antiderivative = 1.85

$$\int \frac{1}{(a+a\csc(x))^{3/2}}\,dx = -\frac{\sqrt{2}\left(\frac{\sin(x)}{\cos(x)+1}\right)^{\frac{3}{2}} - \sqrt{2}\sqrt{\frac{\sin(x)}{\cos(x)+1}}}{2\left(a^{\frac{3}{2}} + \frac{2a^{\frac{3}{2}}\sin(x)}{\cos(x)+1} + \frac{a^{\frac{3}{2}}\sin(x)^2}{(\cos(x)+1)^2}\right)} + \frac{\sqrt{2}\left(\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\sqrt{2}+2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right) + \sqrt{2}\arctan\left(-\frac{1}{2}\sqrt{2}\left(\sqrt{2}-2\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)\right)\right)}{a^{\frac{3}{2}}} - \frac{5\sqrt{2}\arctan\left(\sqrt{\frac{\sin(x)}{\cos(x)+1}}\right)}{2a^{\frac{3}{2}}}$$

input
```
integrate(1/(a+a*csc(x))^(3/2),x, algorithm="maxima")
```

output
```
-1/2*(sqrt(2)*(sin(x)/(cos(x) + 1))^(3/2) - sqrt(2)*sqrt(sin(x)/(cos(x) +
1)))/(a^(3/2) + 2*a^(3/2)*sin(x)/(cos(x) + 1) + a^(3/2)*sin(x)^2/(cos(x) +
 1)^2) + sqrt(2)*(sqrt(2)*arctan(1/2*sqrt(2)*(sqrt(2) + 2*sqrt(sin(x)/(cos
(x) + 1)))) + sqrt(2)*arctan(-1/2*sqrt(2)*(sqrt(2) - 2*sqrt(sin(x)/(cos(x)
 + 1)))))/a^(3/2) - 5/2*sqrt(2)*arctan(sqrt(sin(x)/(cos(x) + 1)))/a^(3/2)
```

3.17.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 243 vs. $2(60) = 120$.

Time = 0.36 (sec) , antiderivative size = 243, normalized size of antiderivative = 3.00

$$
\begin{aligned}
\int \frac{1}{(a+a\csc(x))^{3/2}}\,dx = & -\frac{5\sqrt{2}\arctan\left(\frac{\sqrt{a\tan\left(\frac{1}{2}x\right)}}{\sqrt{a}}\right)}{2a^{\frac{3}{2}}} \\
& + \frac{\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)\arctan\left(\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|}+2\sqrt{a\tan\left(\frac{1}{2}x\right)}\right)}{2\sqrt{|a|}}\right)}{a^3} \\
& + \frac{\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)\arctan\left(-\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|}-2\sqrt{a\tan\left(\frac{1}{2}x\right)}\right)}{2\sqrt{|a|}}\right)}{a^3} \\
& + \frac{\left(a\sqrt{|a|}-|a|^{\frac{3}{2}}\right)\log\left(a\tan\left(\frac{1}{2}x\right)+\sqrt{2}\sqrt{a\tan\left(\frac{1}{2}x\right)}\sqrt{|a|}+|a|\right)}{2a^3} \\
& - \frac{\left(a\sqrt{|a|}-|a|^{\frac{3}{2}}\right)\log\left(a\tan\left(\frac{1}{2}x\right)-\sqrt{2}\sqrt{a\tan\left(\frac{1}{2}x\right)}\sqrt{|a|}+|a|\right)}{2a^3} \\
& - \frac{\sqrt{2}\left(\sqrt{a\tan\left(\frac{1}{2}x\right)}a\tan\left(\frac{1}{2}x\right)-\sqrt{a\tan\left(\frac{1}{2}x\right)}a\right)}{2\left(a\tan\left(\frac{1}{2}x\right)+a\right)^2a}
\end{aligned}
$$

input
```
integrate(1/(a+a*csc(x))^(3/2),x, algorithm="giac")
```

output
```
-5/2*sqrt(2)*arctan(sqrt(a*tan(1/2*x))/sqrt(a))/a^(3/2) + (a*sqrt(abs(a))
+ abs(a)^(3/2))*arctan(1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) + 2*sqrt(a*tan(1/
2*x)))/sqrt(abs(a)))/a^3 + (a*sqrt(abs(a)) + abs(a)^(3/2))*arctan(-1/2*sqr
t(2)*(sqrt(2)*sqrt(abs(a)) - 2*sqrt(a*tan(1/2*x)))/sqrt(abs(a)))/a^3 + 1/2
*(a*sqrt(abs(a)) - abs(a)^(3/2))*log(a*tan(1/2*x) + sqrt(2)*sqrt(a*tan(1/2
*x))*sqrt(abs(a)) + abs(a))/a^3 - 1/2*(a*sqrt(abs(a)) - abs(a)^(3/2))*log(
a*tan(1/2*x) - sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(a))/a^3 - 1/2
*sqrt(2)*(sqrt(a*tan(1/2*x))*a*tan(1/2*x) - sqrt(a*tan(1/2*x))*a)/((a*tan(
1/2*x) + a)^2*a)
```

3.17.9 Mupad [F(-1)]

Timed out.

$$\int \frac{1}{(a+a\csc(x))^{3/2}}\,dx=\int \frac{1}{\left(a+\frac{a}{\sin(x)}\right)^{3/2}}\,dx$$

input
```
int(1/(a + a/sin(x))^(3/2),x)
```

output
```
int(1/(a + a/sin(x))^(3/2), x)
```

3.18 $\int \frac{1}{(a+a\csc(x))^{5/2}}\,dx$

3.18.1 Optimal result

Integrand size = 10, antiderivative size = 100

$$\int \frac{1}{(a+a\csc(x))^{5/2}}\,dx=-\frac{2\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a+a\csc(x)}}\right)}{a^{5/2}}+\frac{43\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a+a\csc(x)}}\right)}{16\sqrt{2}a^{5/2}}+\frac{\cot(x)}{4(a+a\csc(x))^{5/2}}+\frac{11\cot(x)}{16a(a+a\csc(x))^{3/2}}$$

output
```
-2*arctan(cot(x)*a^(1/2)/(a+a*csc(x))^(1/2))/a^(5/2)+1/4*cot(x)/(a+a*csc(x
))^(5/2)+11/16*cot(x)/a/(a+a*csc(x))^(3/2)+43/32*arctan(1/2*cot(x)*a^(1/2)
*2^(1/2)/(a+a*csc(x))^(1/2))/a^(5/2)*2^(1/2)
```

3.18.2 Mathematica [A] (verified)

Time = 0.60 (sec) , antiderivative size = 139, normalized size of antiderivative = 1.39

$$\int \frac{1}{(a+a\csc(x))^{5/2}}\,dx=\frac{\csc^2(x)\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)\left(7+15\cos(2x)-64\arctan\left(\sqrt{-1+\csc(x)}\right)\sqrt{-1+\right.}{32(a(1+}$$

input
```
Integrate[(a + a*Csc[x])^(-5/2),x]
```

output
```
(Csc[x]^2*(Cos[x/2] + Sin[x/2])*(7 + 15*Cos[2*x] - 64*ArcTan[Sqrt[-1 + Csc
[x]]]*Sqrt[-1 + Csc[x]]*(Cos[x/2] + Sin[x/2])^4 + 43*Sqrt[2]*ArcTan[Sqrt[-
1 + Csc[x]]/Sqrt[2]]*Sqrt[-1 + Csc[x]]*(Cos[x/2] + Sin[x/2])^4 + 8*Sin[x])
)/(32*(a*(1 + Csc[x]))^(5/2)*(Cos[x/2] - Sin[x/2]))
```

3.18.3 Rubi [A] (verified)

Time = 0.64 (sec) , antiderivative size = 112, normalized size of antiderivative = 1.12, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.300, Rules used = {3042, 4264, 27, 3042, 4410, 27, 3042, 4408, 3042, 4261, 216, 4282, 216}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{(a\csc(x)+a)^{5/2}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{(a\csc(x)+a)^{5/2}}dx$$

$$\Big\downarrow \text{4264}$$

$$\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}-\frac{\int -\frac{8a-3a\csc(x)}{2(\csc(x)a+a)^{3/2}}dx}{4a^2}$$

$$\Big\downarrow \text{27}$$

$$\frac{\int \frac{8a-3a\csc(x)}{(\csc(x)a+a)^{3/2}}dx}{8a^2}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{8a-3a\csc(x)}{(\csc(x)a+a)^{3/2}}dx}{8a^2}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

$$\Big\downarrow \text{4410}$$

$$\frac{\frac{11a\cot(x)}{2(a\csc(x)+a)^{3/2}}-\frac{\int -\frac{32a^2-11a^2\csc(x)}{2\sqrt{\csc(x)a+a}}dx}{2a^2}}{8a^2}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

$$\Big\downarrow \text{27}$$

$$\frac{\frac{\int \frac{32a^2-11a^2\csc(x)}{\sqrt{\csc(x)a+a}}dx}{4a^2}+\frac{11a\cot(x)}{2(a\csc(x)+a)^{3/2}}}{8a^2}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

$$\Big\downarrow \text{ 3042}$$

$$\frac{\frac{\int \frac{32a^2-11a^2\csc(x)}{\sqrt{\csc(x)a+a}}dx}{4a^2}+\frac{11a\cot(x)}{2(a\csc(x)+a)^{3/2}}}{8a^2}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

$$\Big\downarrow \text{ 4408}$$

$$\frac{\frac{32a\int \sqrt{\csc(x)a+a}dx-43a^2\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx}{4a^2}+\frac{11a\cot(x)}{2(a\csc(x)+a)^{3/2}}}{8a^2}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

$$\Big\downarrow \text{ 3042}$$

$$\frac{\frac{32a\int \sqrt{\csc(x)a+a}dx-43a^2\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx}{4a^2}+\frac{11a\cot(x)}{2(a\csc(x)+a)^{3/2}}}{8a^2}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

$$\Big\downarrow \text{ 4261}$$

$$\frac{\frac{-43a^2\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx-64a^2\int \frac{1}{\frac{a^2\cot^2(x)}{\csc(x)a+a}+a}d\frac{a\cot(x)}{\sqrt{\csc(x)a+a}}}{4a^2}+\frac{11a\cot(x)}{2(a\csc(x)+a)^{3/2}}}{8a^2}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

$$\Big\downarrow \text{ 216}$$

$$\frac{\frac{-43a^2\int \frac{\csc(x)}{\sqrt{\csc(x)a+a}}dx-64a^{3/2}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{4a^2}+\frac{11a\cot(x)}{2(a\csc(x)+a)^{3/2}}}{8a^2}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

$$\Big\downarrow \text{ 4282}$$

$$\frac{\frac{86a^2\int \frac{1}{\frac{a^2\cot^2(x)}{\csc(x)a+a}+2a}d\frac{a\cot(x)}{\sqrt{\csc(x)a+a}}-64a^{3/2}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{4a^2}+\frac{11a\cot(x)}{2(a\csc(x)+a)^{3/2}}}{8a^2}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

$$\Big\downarrow \text{ 216}$$

$$\frac{\frac{43\sqrt{2}a^{3/2}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{2}\sqrt{a\csc(x)+a}}\right)-64a^{3/2}\arctan\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a\csc(x)+a}}\right)}{4a^2}+\frac{11a\cot(x)}{2(a\csc(x)+a)^{3/2}}}{8a^2}+\frac{\cot(x)}{4(a\csc(x)+a)^{5/2}}$$

input
```
Int[(a + a*Csc[x])^(-5/2),x]
```

3.18. $\int \frac{1}{(a+a\csc(x))^{5/2}}\,dx$

output
```
Cot[x]/(4*(a + a*Csc[x])^(5/2)) + ((-64*a^(3/2)*ArcTan[(Sqrt[a]*Cot[x])/Sq
rt[a + a*Csc[x]]] + 43*Sqrt[2]*a^(3/2)*ArcTan[(Sqrt[a]*Cot[x])/(Sqrt[2]*Sq
rt[a + a*Csc[x]])])/(4*a^2) + (11*a*Cot[x])/(2*(a + a*Csc[x])^(3/2)))/(8*a
^2)
```

3.18.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 216
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[b, 2]))*A
rcTan[Rt[b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4261
```
Int[Sqrt[csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_)], x_Symbol] :> Simp[-2*(b/d)
  Subst[Int[1/(a + x^2), x], x, b*(Cot[c + d*x]/Sqrt[a + b*Csc[c + d*x]])],
 x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]
```

rule 4264
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(-Cot[c
+ d*x])*((a + b*Csc[c + d*x])^n/(d*(2*n + 1))), x] + Simp[1/(a^2*(2*n + 1))
   Int[(a + b*Csc[c + d*x])^(n + 1)*(a*(2*n + 1) - b*(n + 1)*Csc[c + d*x]),
 x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0] && LeQ[n, -1] && Int
egerQ[2*n]
```

rule 4282
```
Int[csc[(e_.) + (f_.)*(x_)]/Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)], x_S
ymbol] :> Simp[-2/f   Subst[Int[1/(2*a + x^2), x], x, b*(Cot[e + f*x]/Sqrt[
a + b*Csc[e + f*x]])], x] /; FreeQ[{a, b, e, f}, x] && EqQ[a^2 - b^2, 0]
```

rule 4408
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_
.) + (a_)], x_Symbol] :> Simp[c/a   Int[Sqrt[a + b*Csc[e + f*x]], x], x] -
Simp[(b*c - a*d)/a   Int[Csc[e + f*x]/Sqrt[a + b*Csc[e + f*x]], x], x] /; F
reeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

rule 4410
```
Int[(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_)*(csc[(e_.) + (f_.)*(x_)]*(d
_.) + (c_)), x_Symbol] :> Simp[(-(b*c - a*d))*Cot[e + f*x]*((a + b*Csc[e +
f*x])^m/(b*f*(2*m + 1))), x] + Simp[1/(a^2*(2*m + 1))   Int[(a + b*Csc[e +
f*x])^(m + 1)*Simp[a*c*(2*m + 1) - (b*c - a*d)*(m + 1)*Csc[e + f*x], x], x]
, x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && LtQ[m, -1] &&
EqQ[a^2 - b^2, 0] && IntegerQ[2*m]
```

3.18.4 Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 1103 vs. $2(75) = 150$.

Time = 0.59 (sec) , antiderivative size = 1104, normalized size of antiderivative = 11.04

method	result	size
default	Expression too large to display	1104

input
```
int(1/(a+a*csc(x))^(5/2),x,method=_RETURNVERBOSE)
```

output
```
1/16/(a/(1-cos(x))*(csc(x)*(1-cos(x))^2+2-2*cos(x)+sin(x)))^(5/2)*(csc(x)-
cot(x)+1)*(16*2^(1/2)*ln(-(csc(x)-cot(x)+(csc(x)-cot(x))^(1/2)*2^(1/2)+1)/
((csc(x)-cot(x))^(1/2)*2^(1/2)-csc(x)+cot(x)-1))*(csc(x)-cot(x))+64*2^(1/2
)*arctan((csc(x)-cot(x))^(1/2)*2^(1/2)+1)*(csc(x)-cot(x))+64*2^(1/2)*arcta
n((csc(x)-cot(x))^(1/2)*2^(1/2)-1)*(csc(x)-cot(x))-258*csc(x)^2*arctan((cs
c(x)-cot(x))^(1/2))*(1-cos(x))^2+19*(csc(x)-cot(x))^(3/2)-11*(csc(x)-cot(x
))^(7/2)-43*arctan((csc(x)-cot(x))^(1/2))-19*(csc(x)-cot(x))^(5/2)+16*2^(1
/2)*ln(-((csc(x)-cot(x))^(1/2)*2^(1/2)-csc(x)+cot(x)-1)/(csc(x)-cot(x)+(cs
c(x)-cot(x))^(1/2)*2^(1/2)+1))*(csc(x)-cot(x))-172*csc(x)^3*arctan((csc(x)
-cot(x))^(1/2))*(1-cos(x))^3+24*csc(x)^2*2^(1/2)*ln(-(csc(x)-cot(x)+(csc(x
)-cot(x))^(1/2)*2^(1/2)+1)/((csc(x)-cot(x))^(1/2)*2^(1/2)-csc(x)+cot(x)-1)
)*(1-cos(x))^2+96*csc(x)^2*2^(1/2)*arctan((csc(x)-cot(x))^(1/2)*2^(1/2)+1)
*(1-cos(x))^2+96*csc(x)^2*2^(1/2)*arctan((csc(x)-cot(x))^(1/2)*2^(1/2)-1)*
(1-cos(x))^2+24*csc(x)^2*2^(1/2)*ln(-((csc(x)-cot(x))^(1/2)*2^(1/2)-csc(x)
+cot(x)-1)/(csc(x)-cot(x)+(csc(x)-cot(x))^(1/2)*2^(1/2)+1))*(1-cos(x))^2-1
72*arctan((csc(x)-cot(x))^(1/2))*(csc(x)-cot(x))+64*csc(x)^3*arctan((csc(x
)-cot(x))^(1/2)*2^(1/2)+1)*2^(1/2)*(1-cos(x))^3+64*csc(x)^3*arctan((csc(x)
-cot(x))^(1/2)*2^(1/2)-1)*2^(1/2)*(1-cos(x))^3+16*csc(x)^3*ln(-((csc(x)-co
t(x))^(1/2)*2^(1/2)-csc(x)+cot(x)-1)/(csc(x)-cot(x)+(csc(x)-cot(x))^(1/2)*
2^(1/2)+1))*2^(1/2)*(1-cos(x))^3+4*csc(x)^4*ln(-(csc(x)-cot(x)+(csc(x)-...
```

3.18.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 254 vs. $2(75) = 150$.

Time = 0.27 (sec) , antiderivative size = 546, normalized size of antiderivative = 5.46

$$\int \frac{1}{(a+a\csc(x))^{5/2}}\,dx = \left[-\frac{43\sqrt{2}\left(\cos(x)^3+3\cos(x)^2+\left(\cos(x)^2-2\cos(x)-4\right)\sin(x)-2\cos(x)-4\right.}{}\right.$$

input
```
integrate(1/(a+a*csc(x))^(5/2),x, algorithm="fricas")
```

output
```
[-1/32*(43*sqrt(2)*(cos(x)^3 + 3*cos(x)^2 + (cos(x)^2 - 2*cos(x) - 4)*sin(
x) - 2*cos(x) - 4)*sqrt(-a)*log(-(sqrt(2)*sqrt(-a)*sqrt((a*sin(x) + a)/sin
(x))*sin(x) - a*cos(x))/(sin(x) + 1)) + 32*(cos(x)^3 + 3*cos(x)^2 + (cos(x
)^2 - 2*cos(x) - 4)*sin(x) - 2*cos(x) - 4)*sqrt(-a)*log((2*a*cos(x)^2 + 2*
(cos(x)^2 + (cos(x) + 1)*sin(x) - 1)*sqrt(-a)*sqrt((a*sin(x) + a)/sin(x))
+ a*cos(x) - (2*a*cos(x) + a)*sin(x) - a)/(cos(x) + sin(x) + 1)) - 2*(15*c
os(x)^3 + 4*cos(x)^2 - (15*cos(x)^2 + 11*cos(x) - 4)*sin(x) - 15*cos(x) -
4)*sqrt((a*sin(x) + a)/sin(x)))/(a^3*cos(x)^3 + 3*a^3*cos(x)^2 - 2*a^3*cos
(x) - 4*a^3 + (a^3*cos(x)^2 - 2*a^3*cos(x) - 4*a^3)*sin(x)), 1/16*(43*sqrt
(2)*(cos(x)^3 + 3*cos(x)^2 + (cos(x)^2 - 2*cos(x) - 4)*sin(x) - 2*cos(x) -
 4)*sqrt(a)*arctan(sqrt(2)*sqrt(a)*sqrt((a*sin(x) + a)/sin(x))*(cos(x) + 1
)/(a*cos(x) + a*sin(x) + a)) + 32*(cos(x)^3 + 3*cos(x)^2 + (cos(x)^2 - 2*c
os(x) - 4)*sin(x) - 2*cos(x) - 4)*sqrt(a)*arctan(-sqrt(a)*sqrt((a*sin(x) +
 a)/sin(x))*(cos(x) - sin(x) + 1)/(a*cos(x) + a*sin(x) + a)) + (15*cos(x)^
3 + 4*cos(x)^2 - (15*cos(x)^2 + 11*cos(x) - 4)*sin(x) - 15*cos(x) - 4)*sqr
t((a*sin(x) + a)/sin(x)))/(a^3*cos(x)^3 + 3*a^3*cos(x)^2 - 2*a^3*cos(x) -
4*a^3 + (a^3*cos(x)^2 - 2*a^3*cos(x) - 4*a^3)*sin(x))]
```

3.18.6 Sympy [F]

$$\int \frac{1}{(a+a\csc(x))^{5/2}}\,dx = \int \frac{1}{(a\csc(x)+a)^{\frac{5}{2}}}\,dx$$

input
```
integrate(1/(a+a*csc(x))**(5/2),x)
```

output
```
Integral((a*csc(x) + a)**(-5/2), x)
```

3.18.7 Maxima [F]

$$\int \frac{1}{(a+a\csc(x))^{5/2}}\,dx=\int \frac{1}{(a\csc\left(x\right)+a)^{\frac{5}{2}}}\,dx$$

input
```
integrate(1/(a+a*csc(x))^(5/2),x, algorithm="maxima")
```

output
```
integrate((a*csc(x) + a)^(-5/2), x)
```

3.18.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 286 vs. 2(75) = 150.

Time = 0.36 (sec) , antiderivative size = 286, normalized size of antiderivative = 2.86

$$\int \frac{1}{(a+a\csc(x))^{5/2}}\,dx=-\frac{43\sqrt{2}\arctan\left(\frac{\sqrt{a\tan\left(\frac{1}{2}x\right)}}{\sqrt{a}}\right)}{16\,a^{\frac{5}{2}}}+\frac{\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)\arctan\left(\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|}+2\sqrt{a\tan\left(\frac{1}{2}x\right)}\right)}{2\sqrt{|a|}}\right)}{a^4}+\frac{\left(a\sqrt{|a|}+|a|^{\frac{3}{2}}\right)\arctan\left(-\frac{\sqrt{2}\left(\sqrt{2}\sqrt{|a|}-2\sqrt{a\tan\left(\frac{1}{2}x\right)}\right)}{2\sqrt{|a|}}\right)}{a^4}+\frac{\left(a\sqrt{|a|}-|a|^{\frac{3}{2}}\right)\log\left(a\tan\left(\frac{1}{2}x\right)+\sqrt{2}\sqrt{a\tan\left(\frac{1}{2}x\right)}\sqrt{|a|}+|a|\right)}{2\,a^4}-\frac{\left(a\sqrt{|a|}-|a|^{\frac{3}{2}}\right)\log\left(a\tan\left(\frac{1}{2}x\right)-\sqrt{2}\sqrt{a\tan\left(\frac{1}{2}x\right)}\sqrt{|a|}+|a|\right)}{2\,a^4}-\frac{\sqrt{2}\left(11\sqrt{a\tan\left(\frac{1}{2}x\right)}a^3\tan\left(\frac{1}{2}x\right)^3+19\sqrt{a\tan\left(\frac{1}{2}x\right)}a^3\tan\left(\frac{1}{2}x\right)^2-19\sqrt{a\tan\left(\frac{1}{2}x\right)}a^3\tan\left(\frac{1}{2}x\right)-11\sqrt{a}\right.}{16\left(a\tan\left(\frac{1}{2}x\right)+a\right)^4a^2}$$

input
```
integrate(1/(a+a*csc(x))^(5/2),x, algorithm="giac")
```

output
```
-43/16*sqrt(2)*arctan(sqrt(a*tan(1/2*x))/sqrt(a))/a^(5/2) + (a*sqrt(abs(a)
) + abs(a)^(3/2))*arctan(1/2*sqrt(2)*(sqrt(2)*sqrt(abs(a)) + 2*sqrt(a*tan(
1/2*x)))/sqrt(abs(a)))/a^4 + (a*sqrt(abs(a)) + abs(a)^(3/2))*arctan(-1/2*s
qrt(2)*(sqrt(2)*sqrt(abs(a)) - 2*sqrt(a*tan(1/2*x)))/sqrt(abs(a)))/a^4 + 1
/2*(a*sqrt(abs(a)) - abs(a)^(3/2))*log(a*tan(1/2*x) + sqrt(2)*sqrt(a*tan(1
/2*x))*sqrt(abs(a)) + abs(a))/a^4 - 1/2*(a*sqrt(abs(a)) - abs(a)^(3/2))*lo
g(a*tan(1/2*x) - sqrt(2)*sqrt(a*tan(1/2*x))*sqrt(abs(a)) + abs(a))/a^4 - 1
/16*sqrt(2)*(11*sqrt(a*tan(1/2*x))*a^3*tan(1/2*x)^3 + 19*sqrt(a*tan(1/2*x)
)*a^3*tan(1/2*x)^2 - 19*sqrt(a*tan(1/2*x))*a^3*tan(1/2*x) - 11*sqrt(a*tan(
1/2*x))*a^3)/((a*tan(1/2*x) + a)^4*a^2)
```

3.18.9 Mupad [F(-1)]

Timed out.

$$\int \frac{1}{(a+a\csc(x))^{5/2}}\,dx=\int \frac{1}{\left(a+\frac{a}{\sin(x)}\right)^{5/2}}\,dx$$

input
```
int(1/(a + a/sin(x))^(5/2),x)
```

output
```
int(1/(a + a/sin(x))^(5/2), x)
```

3.19 $\int \sqrt{\csc(e+fx)}\sqrt{a+a\csc(e+fx)}\,dx$

3.19.1 Optimal result

Integrand size = 25, antiderivative size = 37

$$\int \sqrt{\csc(e+fx)}\sqrt{a+a\csc(e+fx)}\,dx = -\frac{2\sqrt{a}\,\text{arcsinh}\left(\frac{\sqrt{a}\cot(e+fx)}{\sqrt{a+a\csc(e+fx)}}\right)}{f}$$

output
```
-2*arcsinh(cot(f*x+e)*a^(1/2)/(a+a*csc(f*x+e))^(1/2))*a^(1/2)/f
```

3.19.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 108 vs. 2(37) = 74.

Time = 0.89 (sec) , antiderivative size = 108, normalized size of antiderivative = 2.92

$$\int \sqrt{\csc(e+fx)}\sqrt{a+a\csc(e+fx)}\,dx$$
$$= \frac{2\cot(e+fx)\sqrt{a(1+\csc(e+fx))}\left(\log(1+\csc(e+fx)) - \log\left(\sqrt{\csc(e+fx)} + \csc^{\frac{3}{2}}(e+fx) + \sqrt{\cot^2(\cdots}\right.\right.}{f\sqrt{\cot^2(e+fx)}\sqrt{1+\csc(e+fx)}}$$

input
```
Integrate[Sqrt[Csc[e + f*x]]*Sqrt[a + a*Csc[e + f*x]],x]
```

output
```
(2*Cot[e + f*x]*Sqrt[a*(1 + Csc[e + f*x])]*(Log[1 + Csc[e + f*x]] - Log[Sq
rt[Csc[e + f*x]] + Csc[e + f*x]^(3/2) + Sqrt[Cot[e + f*x]^2]*Sqrt[1 + Csc[
e + f*x]]]))/(f*Sqrt[Cot[e + f*x]^2]*Sqrt[1 + Csc[e + f*x]])
```

3.19.3 Rubi [A] (verified)

Time = 0.24 (sec) , antiderivative size = 37, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.120, Rules used = {3042, 4288, 222}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sqrt{\csc(e+fx)}\sqrt{a\csc(e+fx)+a}\,dx$$

$$\Big\downarrow \text{ 3042}$$

$$\int \sqrt{\csc(e+fx)}\sqrt{a\csc(e+fx)+a}dx$$

$$\Big\downarrow \text{ 4288}$$

$$-\frac{2\int \frac{1}{\sqrt{\frac{a\cot^2(e+fx)}{\csc(e+fx)a+a}+1}}d\frac{a\cot(e+fx)}{\sqrt{\csc(e+fx)a+a}}}{f}$$

$$\Big\downarrow \text{ 222}$$

$$-\frac{2\sqrt{a}\text{arcsinh}\left(\frac{\sqrt{a}\cot(e+fx)}{\sqrt{a\csc(e+fx)+a}}\right)}{f}$$

input
```
Int[Sqrt[Csc[e + f*x]]*Sqrt[a + a*Csc[e + f*x]],x]
```

output
```
(-2*Sqrt[a]*ArcSinh[(Sqrt[a]*Cot[e + f*x])/Sqrt[a + a*Csc[e + f*x]]])/f
```

3.19.3.1 Defintions of rubi rules used

rule 222
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Simp[ArcSinh[Rt[b, 2]*(x/Sqrt
[a])]/Rt[b, 2], x] /; FreeQ[{a, b}, x] && GtQ[a, 0] && PosQ[b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4288
```
Int[Sqrt[csc[(e_.) + (f_.)*(x_)]*(d_.)]*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
+ (a_)], x_Symbol] :> Simp[-2*(a/(b*f))*Sqrt[a*(d/b)]   Subst[Int[1/Sqrt[1
+ x^2/a], x], x, b*(Cot[e + f*x]/Sqrt[a + b*Csc[e + f*x]])], x] /; FreeQ[{a
, b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && GtQ[a*(d/b), 0]
```

3.19.4 Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 97 vs. $2(31) = 62$.

Time = 2.51 (sec) , antiderivative size = 98, normalized size of antiderivative = 2.65

method	result	size
default	$-\frac{\sin(fx+e)\left(\operatorname{arcsinh}(\cot(fx+e)-\csc(fx+e))+\operatorname{arctanh}\left(\frac{\sqrt{2}}{2\sqrt{\frac{1}{1+\cos(fx+e)}}}\right)\right)\sqrt{\csc(fx+e)}\,\sqrt{a(1+\csc(fx+e))}\,\sqrt{2}}{f(\cos(fx+e)+\sin(fx+e)+1)\sqrt{\frac{1}{1+\cos(fx+e)}}}$	98

input
```
int(csc(f*x+e)^(1/2)*(a+a*csc(f*x+e))^(1/2),x,method=_RETURNVERBOSE)
```

output
```
-1/f*sin(f*x+e)*(arcsinh(cot(f*x+e)-csc(f*x+e))+arctanh(1/2*2^(1/2)/(1/(1+
cos(f*x+e)))^(1/2)))*csc(f*x+e)^(1/2)*(a*(1+csc(f*x+e)))^(1/2)*2^(1/2)/(co
s(f*x+e)+sin(f*x+e)+1)/(1/(1+cos(f*x+e)))^(1/2)
```

3.19.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 75 vs. $2(31) = 62$.

Time = 0.28 (sec) , antiderivative size = 283, normalized size of antiderivative = 7.65

$$\int \sqrt{\csc(e+fx)}\sqrt{a+a\csc(e+fx)}\,dx$$

$$= \left[\frac{\sqrt{a}\log\left(\frac{a\cos(fx+e)^3-7a\cos(fx+e)^2-9a\cos(fx+e)+\left(a\cos(fx+e)^2+8a\cos(fx+e)-a\right)\sin(fx+e)+\frac{4\left(\cos(fx+e)^3+3\cos(fx+e)^2-\left(\cos(f\ldots\right.\right.}{\ldots}}{\cos(fx+e)^3+\cos(fx+e)^2+\left(\cos(fx+e)^2-1\right)\sin(fx+e)-\cos(fx+e\ldots}\right.\ldots}{2f} \right.$$

input
```
integrate(csc(f*x+e)^(1/2)*(a+a*csc(f*x+e))^(1/2),x, algorithm="fricas")
```

output
```
[1/2*sqrt(a)*log((a*cos(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 9*a*cos(f*x + e)
 + (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)*sin(f*x + e) + 4*(cos(f*x + e
)^3 + 3*cos(f*x + e)^2 - (cos(f*x + e)^2 - 2*cos(f*x + e) - 3)*sin(f*x + e
) - cos(f*x + e) - 3)*sqrt(a)*sqrt((a*sin(f*x + e) + a)/sin(f*x + e))/sqrt
(sin(f*x + e)) - a)/(cos(f*x + e)^3 + cos(f*x + e)^2 + (cos(f*x + e)^2 - 1
)*sin(f*x + e) - cos(f*x + e) - 1))/f, sqrt(-a)*arctan(1/2*(cos(f*x + e)^2
 + 2*sin(f*x + e) - 1)*sqrt(-a)*sqrt((a*sin(f*x + e) + a)/sin(f*x + e))/(a
*cos(f*x + e)*sqrt(sin(f*x + e))))/f]
```

3.19.6 Sympy [F]

$$\int \sqrt{\csc(e+fx)}\sqrt{a+a\csc(e+fx)}\,dx=\int \sqrt{a\left(\csc\left(e+fx\right)+1\right)}\sqrt{\csc\left(e+fx\right)}\,dx$$

input
```
integrate(csc(f*x+e)**(1/2)*(a+a*csc(f*x+e))**(1/2),x)
```

output
```
Integral(sqrt(a*(csc(e + f*x) + 1))*sqrt(csc(e + f*x)), x)
```

3.19.7 Maxima [F]

$$\int \sqrt{\csc(e+fx)}\sqrt{a+a\csc(e+fx)}\,dx=\int \sqrt{a\csc\left(fx+e\right)+a}\sqrt{\csc\left(fx+e\right)}\,dx$$

input
```
integrate(csc(f*x+e)^(1/2)*(a+a*csc(f*x+e))^(1/2),x, algorithm="maxima")
```

output
```
integrate(sqrt(a*csc(f*x + e) + a)*sqrt(csc(f*x + e)), x)
```

3.19.8 Giac [F(-2)]

Exception generated.

$$\int \sqrt{\csc(e+fx)}\sqrt{a+a\csc(e+fx)}\,dx = \text{Exception raised: TypeError}$$

input
```
integrate(csc(f*x+e)^(1/2)*(a+a*csc(f*x+e))^(1/2),x, algorithm="giac")
```

output
```
Exception raised: TypeError >> an error occurred running a Giac command:IN
PUT:sage2:=int(sage0,sageVARx):;OUTPUT:Limit: Max order reached or unable
to make series expansion Error: Bad Argument Value
```

3.19.9 Mupad [F(-1)]

Timed out.

$$\int \sqrt{\csc(e+fx)}\sqrt{a+a\csc(e+fx)}\,dx = \int \sqrt{a+\frac{a}{\sin(e+fx)}}\sqrt{\frac{1}{\sin(e+fx)}}\,dx$$

input
```
int((a + a/sin(e + f*x))^(1/2)*(1/sin(e + f*x))^(1/2),x)
```

output
```
int((a + a/sin(e + f*x))^(1/2)*(1/sin(e + f*x))^(1/2), x)
```

3.20 $\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\,dx$

3.20.1 Optimal result

Integrand size = 28, antiderivative size = 38

$$\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\,dx=-\frac{2\sqrt{a}\operatorname{arcsinh}\left(\frac{\sqrt{a}\cot(e+fx)}{\sqrt{a-a\csc(e+fx)}}\right)}{f}$$

output
```
-2*arcsinh(cot(f*x+e)*a^(1/2)/(a-a*csc(f*x+e))^(1/2))*a^(1/2)/f
```

3.20.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 101 vs. 2(38) = 76.

Time = 3.14 (sec) , antiderivative size = 101, normalized size of antiderivative = 2.66

$$\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\,dx$$
$$=\frac{2\left(\operatorname{arcsinh}\left(\tan\left(\frac{1}{2}(e+fx)\right)\right)+\operatorname{arctanh}\left(\sqrt{\sec^2\left(\frac{1}{2}(e+fx)\right)}\right)\right)\sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\tan\left(\frac{1}{2}\right.}{f\sqrt{\sec^2\left(\frac{1}{2}(e+fx)\right)}\left(-1+\tan\left(\frac{1}{2}(e+fx)\right)\right)}$$

input
```
Integrate[Sqrt[-Csc[e + f*x]]*Sqrt[a - a*Csc[e + f*x]],x]
```

output
```
(2*(ArcSinh[Tan[(e + f*x)/2]] + ArcTanh[Sqrt[Sec[(e + f*x)/2]^2]])*Sqrt[-C
sc[e + f*x]]*Sqrt[a - a*Csc[e + f*x]]*Tan[(e + f*x)/2])/(f*Sqrt[Sec[(e + f
*x)/2]^2]*(-1 + Tan[(e + f*x)/2]))
```

3.20.3 Rubi [A] (verified)

Time = 0.24 (sec) , antiderivative size = 38, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.107$, Rules used = {3042, 4288, 222}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}dx$$

$$\Big\downarrow \text{4288}$$

$$\frac{2\int \frac{1}{\sqrt{\frac{a\cot^2(e+fx)}{a-a\csc(e+fx)}+1}}d\left(-\frac{a\cot(e+fx)}{\sqrt{a-a\csc(e+fx)}}\right)}{f}$$

$$\Big\downarrow \text{222}$$

$$-\frac{2\sqrt{a}\text{arcsinh}\left(\frac{\sqrt{a}\cot(e+fx)}{\sqrt{a-a\csc(e+fx)}}\right)}{f}$$

input
```
Int[Sqrt[-Csc[e + f*x]]*Sqrt[a - a*Csc[e + f*x]],x]
```

output
```
(-2*Sqrt[a]*ArcSinh[(Sqrt[a]*Cot[e + f*x])/Sqrt[a - a*Csc[e + f*x]]])/f
```

3.20.3.1 Defintions of rubi rules used

rule 222
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Simp[ArcSinh[Rt[b, 2]*(x/Sqrt
[a])]/Rt[b, 2], x] /; FreeQ[{a, b}, x] && GtQ[a, 0] && PosQ[b]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4288
```
Int[Sqrt[csc[(e_.) + (f_.)*(x_)]*(d_.)]*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
+ (a_)], x_Symbol] :> Simp[-2*(a/(b*f))*Sqrt[a*(d/b)]   Subst[Int[1/Sqrt[1
+ x^2/a], x], x, b*(Cot[e + f*x]/Sqrt[a + b*Csc[e + f*x]])], x] /; FreeQ[{a
, b, d, e, f}, x] && EqQ[a^2 - b^2, 0] && GtQ[a*(d/b), 0]
```

3.20.4 Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 126 vs. $2(32) = 64$.

Time = 2.75 (sec) , antiderivative size = 127, normalized size of antiderivative = 3.34

method	result
default	$-\frac{\sin(fx+e)\left(\arctan\left(\frac{\sqrt{2}}{2\sqrt{-\frac{1}{1+\cos(fx+e)}}}\right)+\arctan\left(\frac{\sqrt{2}\,\sin(fx+e)}{2(1+\cos(fx+e))\sqrt{-\frac{1}{1+\cos(fx+e)}}}\right)\right)\sqrt{-a(-1+\csc(fx+e))}\,\sqrt{-\csc(fx+e)}\,\sqrt{2}}{f(-\cos(fx+e)+\sin(fx+e)-1)\sqrt{-\frac{1}{1+\cos(fx+e)}}}$

input
```
int((-csc(f*x+e))^(1/2)*(a-a*csc(f*x+e))^(1/2),x,method=_RETURNVERBOSE)
```

output
```
-1/f*sin(f*x+e)*(arctan(1/2*2^(1/2)/(-1/(1+cos(f*x+e)))^(1/2))+arctan(1/2*
2^(1/2)*sin(f*x+e)/(1+cos(f*x+e))/(-1/(1+cos(f*x+e)))^(1/2)))*(-a*(-1+csc(
f*x+e)))^(1/2)*(-csc(f*x+e))^(1/2)*2^(1/2)/(-cos(f*x+e)+sin(f*x+e)-1)/(-1/
(1+cos(f*x+e)))^(1/2)
```

3.20.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 81 vs. $2(32) = 64$.

Time = 0.26 (sec) , antiderivative size = 296, normalized size of antiderivative = 7.79

$$\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\,dx$$

$$= \left[\frac{\sqrt{a}\log\left(\frac{a\cos(fx+e)^3-7\,a\cos(fx+e)^2-4\left(\cos(fx+e)^3+3\,\cos(fx+e)^2+\left(\cos(fx+e)^2-2\,\cos(fx+e)-3\right)\sin(fx+e)-\cos(fx+e)-3\right)\sqrt{a}\sqrt{\cdots}}{\cos(fx+e)^3+\cos(fx+e)^2-\left(\cos(fx+e)^2-1\right)\sin(fx+\cdots}\right.}{2\,f} \right.$$

input
```
integrate((-csc(f*x+e))^(1/2)*(a-a*csc(f*x+e))^(1/2),x, algorithm="fricas"
)
```

output
```
[1/2*sqrt(a)*log((a*cos(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 4*(cos(f*x + e)^
3 + 3*cos(f*x + e)^2 + (cos(f*x + e)^2 - 2*cos(f*x + e) - 3)*sin(f*x + e)
- cos(f*x + e) - 3)*sqrt(a)*sqrt((a*sin(f*x + e) - a)/sin(f*x + e))*sqrt(-
1/sin(f*x + e)) - 9*a*cos(f*x + e) - (a*cos(f*x + e)^2 + 8*a*cos(f*x + e)
- a)*sin(f*x + e) - a)/(cos(f*x + e)^3 + cos(f*x + e)^2 - (cos(f*x + e)^2
- 1)*sin(f*x + e) - cos(f*x + e) - 1))/f, sqrt(-a)*arctan(-1/2*(cos(f*x +
e)^2 - 2*sin(f*x + e) - 1)*sqrt(-a)*sqrt((a*sin(f*x + e) - a)/sin(f*x + e)
)*sqrt(-1/sin(f*x + e))/(a*cos(f*x + e)))/f]
```

3.20.6 Sympy [F]

$$\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\,dx = \int \sqrt{-\csc{(e+fx)}}\sqrt{-a\left(\csc{(e+fx)}-1\right)}\,dx$$

input
```
integrate((-csc(f*x+e))**(1/2)*(a-a*csc(f*x+e))**(1/2),x)
```

output
```
Integral(sqrt(-csc(e + f*x))*sqrt(-a*(csc(e + f*x) - 1)), x)
```

3.20.7 Maxima [F]

$$\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\,dx = \int \sqrt{-a\csc{(fx+e)}+a}\sqrt{-\csc{(fx+e)}}\,dx$$

input
```
integrate((-csc(f*x+e))^(1/2)*(a-a*csc(f*x+e))^(1/2),x, algorithm="maxima"
)
```

output
```
integrate(sqrt(-a*csc(f*x + e) + a)*sqrt(-csc(f*x + e)), x)
```

3.20.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 101 vs. 2(32) = 64.

Time = 0.66 (sec) , antiderivative size = 101, normalized size of antiderivative = 2.66

$$\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\,dx = -\frac{\frac{2\,a\arctan\left(\frac{a^{\frac{3}{2}}\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)+\sqrt{a^{3}\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^{2}+a^{3}}}{\sqrt{-a}a}\right)}{\sqrt{-a}} - \sqrt{a}\log\left(\left|a^{\frac{3}{2}}\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)+\sqrt{a^{3}\tan\left(\frac{1}{2}\,fx+\frac{1}{2}\,e\right)^{2}+a^{3}}\right|\right)}{f}$$

input
```
integrate((-csc(f*x+e))^(1/2)*(a-a*csc(f*x+e))^(1/2),x, algorithm="giac")
```

output
```
-(2*a*arctan((a^(3/2)*tan(1/2*f*x + 1/2*e) + sqrt(a^3*tan(1/2*f*x + 1/2*e)
^2 + a^3))/(sqrt(-a)*a))/sqrt(-a) - sqrt(a)*log(abs(a^(3/2)*tan(1/2*f*x +
1/2*e) + sqrt(a^3*tan(1/2*f*x + 1/2*e)^2 + a^3))))/f
```

3.20.9 Mupad [F(-1)]

Timed out.

$$\int \sqrt{-\csc(e+fx)}\sqrt{a-a\csc(e+fx)}\,dx = \int \sqrt{a-\frac{a}{\sin(e+f\,x)}}\,\sqrt{-\frac{1}{\sin(e+f\,x)}}\,dx$$

input
```
int((a - a/sin(e + f*x))^(1/2)*(-1/sin(e + f*x))^(1/2),x)
```

output
```
int((a - a/sin(e + f*x))^(1/2)*(-1/sin(e + f*x))^(1/2), x)
```

3.21 $\int \csc^{\frac{4}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$

3.21.1 Optimal result

Integrand size = 25, antiderivative size = 254

$$\int \csc^{\frac{4}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = -\frac{6a\cos(c+dx)\csc^{\frac{4}{3}}(c+dx)}{5d\sqrt{a+a\csc(c+dx)}}$$

$$-\frac{4\,3^{3/4}\sqrt{2+\sqrt{3}}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\csc^{\frac{2}{3}}(c+dx)}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}\,\text{EllipticF}\left(\arcsin\left(\frac{1-}{1+}\right.\right.}{5d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}(a-a\csc(c+dx))\sqrt{a+a\csc(c+dx)}}$$

output
```
-6/5*a*cos(d*x+c)*csc(d*x+c)^(4/3)/d/(a+a*csc(d*x+c))^(1/2)-4/5*3^(3/4)*a^
2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(1/2))/(
1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2)+1/2*2^(1/2))*((1+c
sc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/
(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c
)^(1/3)+3^(1/2))^2)^(1/2)
```

3.21.2 Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 4 in optimal.

Time = 8.59 (sec) , antiderivative size = 102, normalized size of antiderivative = 0.40

$$\int \csc^{\frac{4}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = -\frac{2\sqrt{a(1+\csc(c+dx))}\left(3\sqrt[3]{\csc(c+dx)}+2\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{2}{3},\frac{3}{2},1-\csc(c+dx)\right)\right)\left(\cos\left(\frac{1}{2}(c+\right.\right.}{5d\left(\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)\right)}$$

input
```
Integrate[Csc[c + d*x]^(4/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

output
```
(-2*Sqrt[a*(1 + Csc[c + d*x])]*(3*Csc[c + d*x]^(1/3) + 2*Hypergeometric2F1
[1/2, 2/3, 3/2, 1 - Csc[c + d*x]])*(Cos[(c + d*x)/2] - Sin[(c + d*x)/2]))/
(5*d*(Cos[(c + d*x)/2] + Sin[(c + d*x)/2]))
```

3.21.3 Rubi [A] (verified)

Time = 0.35 (sec) , antiderivative size = 266, normalized size of antiderivative = 1.05, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {3042, 4293, 60, 73, 759}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^{\frac{4}{3}}(c+dx)\sqrt{a\csc(c+dx)+a}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \csc(c+dx)^{4/3}\sqrt{a\csc(c+dx)+a}dx$$

$$\Big\downarrow \text{4293}$$

$$\frac{a^2\cot(c+dx)\int \frac{\sqrt[3]{\csc(c+dx)}}{\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \text{60}$$

$$\frac{a^2\cot(c+dx)\left(\frac{2}{5}\int \frac{1}{\csc^{\frac{2}{3}}(c+dx)\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)-\frac{6\sqrt[3]{\csc(c+dx)}\sqrt{a-a\csc(c+dx)}}{5a}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \ 73$$

$$\frac{a^2\cot(c+dx)\left(\frac{6}{5}\int \frac{1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\frac{6\sqrt[3]{\csc(c+dx)}\sqrt{a-a\csc(c+dx)}}{5a}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \ 759$$

$$\frac{a^2\cot(c+dx)\left(-\frac{4\,3^{3/4}\sqrt{2+\sqrt{3}}\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\text{EllipticF}\left(\arcsin\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right.\right.}{5\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\sqrt{a-a\csc(c+dx)}}\right.}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

input
```
Int[Csc[c + d*x]^(4/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

output
```
(a^2*Cot[c + d*x]*((-6*Csc[c + d*x]^(1/3)*Sqrt[a - a*Csc[c + d*x]])/(5*a)
- (4*3^(3/4)*Sqrt[2 + Sqrt[3]]*(1 - Csc[c + d*x]^(1/3))*Sqrt[(1 + Csc[c +
d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*Ell
ipticF[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*
x]^(1/3))], -7 - 4*Sqrt[3]])/(5*Sqrt[(1 - Csc[c + d*x]^(1/3))/(1 + Sqrt[3]
 - Csc[c + d*x]^(1/3))^2]*Sqrt[a - a*Csc[c + d*x]])))/(d*Sqrt[a - a*Csc[c
+ d*x]]*Sqrt[a + a*Csc[c + d*x]])
```

3.21.3.1 Defintions of rubi rules used

rule 60
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Simp[n*((b*c - a*d)/(
b*(m + n + 1)))   Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
 c, d}, x] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ[m, 0] && ( !Integer
Q[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n + 2, 0] && IntLinear
Q[a, b, c, d, m, n, x]
```

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 759
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[2*Sqrt[2 + Sqrt[3]]*(s + r*x)*(Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]/(3^(1/4)*r*Sqrt[a + b*x^3]*Sqrt[s*
((s + r*x)/((1 + Sqrt[3])*s + r*x)^2)]))*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]], x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4293
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x]]
*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
 Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

3.21.4 Maple [F]

$$\int \csc (dx + c)^{\frac{4}{3}} \sqrt{a + a \csc (dx + c)}dx$$

input
```
int(csc(d*x+c)^(4/3)*(a+a*csc(d*x+c))^(1/2),x)
```

output
```
int(csc(d*x+c)^(4/3)*(a+a*csc(d*x+c))^(1/2),x)
```

3.21.5 Fricas [F]

$$\int \csc^{\frac{4}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a\csc(dx+c)+a}\csc(dx+c)^{\frac{4}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(4/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="fricas")
```

output
```
integral(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(4/3), x)
```

3.21.6 Sympy [F(-1)]

Timed out.

$$\int \csc^{\frac{4}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \text{Timed out}$$

input
```
integrate(csc(d*x+c)**(4/3)*(a+a*csc(d*x+c))**(1/2),x)
```

output
```
Timed out
```

3.21.7 Maxima [F]

$$\int \csc^{\frac{4}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a\csc(dx+c)+a}\csc(dx+c)^{\frac{4}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(4/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="maxima")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(4/3), x)
```

3.21.8 Giac [F]

$$\int \csc^{\frac{4}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx=\int \sqrt{a\csc\left(dx+c\right)+a}\csc\left(dx+c\right)^{\frac{4}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(4/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="giac")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(4/3), x)
```

3.21.9 Mupad [F(-1)]

Timed out.

$$\int \csc^{\frac{4}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx=\int \sqrt{a+\frac{a}{\sin\left(c+d\,x\right)}}\,{\left(\frac{1}{\sin\left(c+d\,x\right)}\right)}^{4/3}\,dx$$

input
```
int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(4/3),x)
```

output
```
int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(4/3), x)
```

3.22 $\int \sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}\,dx$

3.22.1 Optimal result

Integrand size = 25, antiderivative size = 213

$$\int \sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}\,dx = -\frac{2\,3^{3/4}\sqrt{2+\sqrt{3}}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\csc^{\frac{2}{3}}(c+dx)}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}\,\text{EllipticF}\left(\arcsin\left(\frac{1-\cdots}{1+\cdots}\right.\right.}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}(a-a\csc(c+dx))\sqrt{a+a\csc(c+dx)}}$$

output
```
-2*3^(3/4)*a^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/
3)-3^(1/2))/(1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2)+1/2*2
^(1/2))*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2)
)^2)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))
/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)
```

3.22.2 Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 4 in optimal.

Time = 3.08 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.22

$$\int \sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}\,dx$$
$$= -\frac{2a\cot(c+dx)\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{2}{3},\frac{3}{2},1-\csc(c+dx)\right)}{d\sqrt{a(1+\csc(c+dx))}}$$

input
```
Integrate[Csc[c + d*x]^(1/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

output
```
(-2*a*Cot[c + d*x]*Hypergeometric2F1[1/2, 2/3, 3/2, 1 - Csc[c + d*x]])/(d*
Sqrt[a*(1 + Csc[c + d*x])])
```

3.22.3 Rubi [A] (verified)

Time = 0.30 (sec) , antiderivative size = 213, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.160, Rules used = {3042, 4293, 73, 759}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sqrt[3]{\csc(c+dx)}\sqrt{a\csc(c+dx)+a}\,dx$$
$$\Big\downarrow \text{3042}$$
$$\int \sqrt[3]{\csc(c+dx)}\sqrt{a\csc(c+dx)+a}dx$$
$$\Big\downarrow \text{4293}$$
$$\frac{a^2\cot(c+dx)\int \frac{1}{\csc^{\frac{2}{3}}(c+dx)\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$
$$\Big\downarrow \text{73}$$
$$\frac{3a^2\cot(c+dx)\int \frac{1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$
$$\Big\downarrow \text{759}$$

$$-\frac{2\,3^{3/4}\sqrt{2+\sqrt{3}}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\operatorname{EllipticF}\left(\arcsin\left(\frac{-\sqrt[3]{\csc(c+\cdots}}{-\sqrt[3]{\csc(c+\cdots}}\right.\right.}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}(a-a\csc(c+dx))\sqrt{a\csc(c+dx)+a}}$$

input
```
Int[Csc[c + d*x]^(1/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

output
```
(-2*3^(3/4)*Sqrt[2 + Sqrt[3]]*a^2*Cot[c + d*x]*(1 - Csc[c + d*x]^(1/3))*Sq
rt[(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*
x]^(1/3))^2]*EllipticF[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt
[3] - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(d*Sqrt[(1 - Csc[c + d*x]^(1/
3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*(a - a*Csc[c + d*x])*Sqrt[a + a*
Csc[c + d*x]])
```

3.22.3.1 Defintions of rubi rules used

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 759
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[2*Sqrt[2 + Sqrt[3]]*(s + r*x)*(Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]/(3^(1/4)*r*Sqrt[a + b*x^3]*Sqrt[s*
((s + r*x)/((1 + Sqrt[3])*s + r*x)^2)]))*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]], x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4293
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x]]
*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
 Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

3.22.4 Maple [F]

$$\int \csc \left(dx+c\right)^{\frac{1}{3}} \sqrt{a+a\csc \left(dx+c\right)}dx$$

input
```
int(csc(d*x+c)^(1/3)*(a+a*csc(d*x+c))^(1/2),x)
```

output
```
int(csc(d*x+c)^(1/3)*(a+a*csc(d*x+c))^(1/2),x)
```

3.22.5 Fricas [F]

$$\int \sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}\,dx=\int \sqrt{a\csc \left(dx+c\right)+a}\csc \left(dx+c\right)^{\frac{1}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(1/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="fricas")
```

output
```
integral(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(1/3), x)
```

3.22.6 Sympy [F]

$$\int \sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}\,dx=\int \sqrt{a\left(\csc \left(c+dx\right)+1\right)}\sqrt[3]{\csc \left(c+dx\right)}\,dx$$

input
```
integrate(csc(d*x+c)**(1/3)*(a+a*csc(d*x+c))**(1/2),x)
```

output
```
Integral(sqrt(a*(csc(c + d*x) + 1))*csc(c + d*x)**(1/3), x)
```

3.22.7 Maxima [F]

$$\int \sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a\csc(dx+c)+a}\csc(dx+c)^{\frac{1}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(1/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="maxima")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(1/3), x)
```

3.22.8 Giac [F]

$$\int \sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a\csc(dx+c)+a}\csc(dx+c)^{\frac{1}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(1/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="giac")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(1/3), x)
```

3.22.9 Mupad [F(-1)]

Timed out.

$$\int \sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a+\frac{a}{\sin(c+d\,x)}}\left(\frac{1}{\sin(c+d\,x)}\right)^{1/3}\,dx$$

input
```
int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(1/3),x)
```

output
```
int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(1/3), x)
```

3.23 $\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{2}{3}}(c+dx)}\,dx$

3.23.1 Optimal result

Integrand size = 25, antiderivative size = 254

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{2}{3}}(c+dx)}\,dx = -\frac{3a\cos(c+dx)\sqrt[3]{\csc(c+dx)}}{2d\sqrt{a+a\csc(c+dx)}}$$
$$-\frac{3^{3/4}\sqrt{2+\sqrt{3}}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\csc^{\frac{2}{3}}(c+dx)}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}\,\text{EllipticF}\left(\arcsin\left(\frac{1-\sqrt{3}\cdots}{1+\sqrt{3}\cdots}\right.\right.}{2d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}(a-a\csc(c+dx))\sqrt{a+a\csc(c+dx)}}$$

output
```
-3/2*a*cos(d*x+c)*csc(d*x+c)^(1/3)/d/(a+a*csc(d*x+c))^(1/2)-1/2*3^(3/4)*a^
2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(1/2))/(
1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2)+1/2*2^(1/2))*((1+c
sc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/
(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c
)^(1/3)+3^(1/2))^2)^(1/2)
```

3.23.2 Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 4 in optimal.

Time = 3.38 (sec) , antiderivative size = 110, normalized size of antiderivative = 0.43

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{2}{3}}(c+dx)}\,dx = -\frac{\sqrt{a(1+\csc(c+dx))}\left(3+\csc^{\frac{2}{3}}(c+dx)\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{2}{3},\frac{3}{2},1-\csc(c+dx)\right)\right)\left(\cos\left(\frac{1}{2}(c+dx\right.\right.}{2d\csc^{\frac{2}{3}}(c+dx)\left(\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)\right)}$$

input
```
Integrate[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(2/3),x]
```

output
```
-1/2*(Sqrt[a*(1 + Csc[c + d*x])]*(3 + Csc[c + d*x]^(2/3)*Hypergeometric2F1
[1/2, 2/3, 3/2, 1 - Csc[c + d*x]])*(Cos[(c + d*x)/2] - Sin[(c + d*x)/2]))/
(d*Csc[c + d*x]^(2/3)*(Cos[(c + d*x)/2] + Sin[(c + d*x)/2]))
```

3.23.3 Rubi [A] (verified)

Time = 0.32 (sec) , antiderivative size = 266, normalized size of antiderivative = 1.05, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.200, Rules used = {3042, 4293, 61, 73, 759}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{a\csc(c+dx)+a}}{\csc^{\frac{2}{3}}(c+dx)}dx$$

$$\Big\downarrow \text{ 3042}$$

$$\int \frac{\sqrt{a\csc(c+dx)+a}}{\csc(c+dx)^{2/3}}dx$$

$$\Big\downarrow \text{ 4293}$$

$$\frac{a^2\cot(c+dx)\int \frac{1}{\csc^{\frac{5}{3}}(c+dx)\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \text{ 61}$$

$$\frac{a^2\cot(c+dx)\left(\frac{1}{4}\int\frac{1}{\csc^{\frac{2}{3}}(c+dx)\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)-\frac{3\sqrt{a-a\csc(c+dx)}}{2a\csc^{\frac{2}{3}}(c+dx)}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \text{73}$$

$$\frac{a^2\cot(c+dx)\left(\frac{3}{4}\int\frac{1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\frac{3\sqrt{a-a\csc(c+dx)}}{2a\csc^{\frac{2}{3}}(c+dx)}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \text{759}$$

$$\frac{a^2\cot(c+dx)\left(-\frac{3^{3/4}\sqrt{2+\sqrt{3}}\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\text{EllipticF}\left(\arcsin\left(\frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1}\right)\right.}{2\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\sqrt{a-a\csc(c+dx)}}\right.}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

input
```
Int[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(2/3),x]
```

output
```
(a^2*Cot[c + d*x]*((-3*Sqrt[a - a*Csc[c + d*x]])/(2*a*Csc[c + d*x]^(2/3))
- (3^(3/4)*Sqrt[2 + Sqrt[3]]*(1 - Csc[c + d*x]^(1/3))*Sqrt[(1 + Csc[c + d*
x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*Ellip
ticF[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]
^(1/3))], -7 - 4*Sqrt[3]])/(2*Sqrt[(1 - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] -
 Csc[c + d*x]^(1/3))^2]*Sqrt[a - a*Csc[c + d*x]])))/(d*Sqrt[a - a*Csc[c +
d*x]]*Sqrt[a + a*Csc[c + d*x]])
```

3.23.3.1 Defintions of rubi rules used

rule 61
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^(n + 1)/((b*c - a*d)*(m + 1))), x] - Simp[d*((
m + n + 2)/((b*c - a*d)*(m + 1)))   Int[(a + b*x)^(m + 1)*(c + d*x)^n, x],
x] /; FreeQ[{a, b, c, d, n}, x] && LtQ[m, -1] &&  !(LtQ[n, -1] && (EqQ[a, 0
] || (NeQ[c, 0] && LtQ[m - n, 0] && IntegerQ[n]))) && IntLinearQ[a, b, c, d
, m, n, x]
```

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 759
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[2*Sqrt[2 + Sqrt[3]]*(s + r*x)*(Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]/(3^(1/4)*r*Sqrt[a + b*x^3]*Sqrt[s*
((s + r*x)/((1 + Sqrt[3])*s + r*x)^2)]))*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]], x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4293
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x]]
*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
 Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

3.23.4 Maple [F]

$$\int \frac{\sqrt{a + a\csc(dx + c)}}{\csc(dx + c)^{\frac{2}{3}}}dx$$

input
```
int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(2/3),x)
```

output
```
int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(2/3),x)
```

3.23.5 Fricas [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{2}{3}}(c+dx)}\,dx = \int \frac{\sqrt{a\csc\left(dx+c\right)+a}}{\csc\left(dx+c\right)^{\frac{2}{3}}}\,dx$$

input
```
integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(2/3),x, algorithm="fricas")
```

output
```
integral(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(2/3), x)
```

3.23.6 Sympy [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{2}{3}}(c+dx)}\,dx = \int \frac{\sqrt{a\left(\csc\left(c+dx\right)+1\right)}}{\csc^{\frac{2}{3}}\left(c+dx\right)}\,dx$$

input
```
integrate((a+a*csc(d*x+c))**(1/2)/csc(d*x+c)**(2/3),x)
```

output
```
Integral(sqrt(a*(csc(c + d*x) + 1))/csc(c + d*x)**(2/3), x)
```

3.23.7 Maxima [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{2}{3}}(c+dx)}\,dx = \int \frac{\sqrt{a\csc\left(dx+c\right)+a}}{\csc\left(dx+c\right)^{\frac{2}{3}}}\,dx$$

input
```
integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(2/3),x, algorithm="maxima")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(2/3), x)
```

3.23.8 Giac [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{2}{3}}(c+dx)}\,dx=\int \frac{\sqrt{a\csc(dx+c)+a}}{\csc(dx+c)^{\frac{2}{3}}}\,dx$$

input
```
integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(2/3),x, algorithm="giac")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(2/3), x)
```

3.23.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{2}{3}}(c+dx)}\,dx=\int \frac{\sqrt{a+\frac{a}{\sin(c+d\,x)}}}{\left(\frac{1}{\sin(c+d\,x)}\right)^{2/3}}\,dx$$

input
```
int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(2/3),x)
```

output
```
int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(2/3), x)
```

3.24 $\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$

3.24.1 Optimal result

Integrand size = 25, antiderivative size = 514

$$\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$$

$$= \frac{24a\cot(c+dx)}{7d\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)\sqrt{a+a\csc(c+dx)}} - \frac{6a\cos(c+dx)\csc^{\frac{5}{3}}(c+dx)}{7d\sqrt{a+a\csc(c+dx)}}$$

$$- \frac{12\sqrt[4]{3}\sqrt{2-\sqrt{3}}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\csc^{\frac{2}{3}}(c+dx)}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}E\left(\arcsin\left(\frac{1-\sqrt{3}-\sqrt[3]{cs}}{1+\sqrt{3}-\sqrt[3]{cs}}\right.\right.}{7d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}(a-a\csc(c+dx))\sqrt{a+a\csc(c+dx)}}$$

$$+ \frac{8\sqrt{2}3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\csc^{\frac{2}{3}}(c+dx)}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}\text{EllipticF}\left(\arcsin\left(\frac{1-\sqrt{3}-\sqrt[3]{c}}{1+\sqrt{3}-\sqrt[3]{c}}\right.\right.}{7d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}(a-a\csc(c+dx))\sqrt{a+a\csc(c+dx)}}$$

output
```
-6/7*a*cos(d*x+c)*csc(d*x+c)^(5/3)/d/(a+a*csc(d*x+c))^(1/2)+24/7*a*cot(d*x
+c)/d/(1-csc(d*x+c)^(1/3)+3^(1/2))/(a+a*csc(d*x+c))^(1/2)+8/7*3^(3/4)*a^2*
cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(1/2))/(1-
csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*2^(1/2)*((1+csc(d*x+c)^(1/3)+csc(
d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/(a-a*csc(d*x+c))/(a+
a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^
(1/2)-12/7*3^(1/4)*a^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticE((1-csc(d*
x+c)^(1/3)-3^(1/2))/(1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/
2)-1/2*2^(1/2))*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)
+3^(1/2))^2)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c
)^(1/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)
```

3.24.2 Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 4 in optimal.

Time = 21.35 (sec) , antiderivative size = 120, normalized size of antiderivative = 0.23

$$\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx =$$

$$-\frac{2\sqrt{a(1+\csc(c+dx))}\left(3(4+\csc(c+dx))-8\sqrt[3]{\csc(c+dx)}\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{4}{3},\frac{3}{2},1-\csc(c+d\right.\right.}{7d\sqrt[3]{\csc(c+dx)}\left(\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)\right)}$$

input
```
Integrate[Csc[c + d*x]^(5/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

output
```
(-2*Sqrt[a*(1 + Csc[c + d*x])]*(3*(4 + Csc[c + d*x]) - 8*Csc[c + d*x]^(1/3
)*Hypergeometric2F1[1/2, 4/3, 3/2, 1 - Csc[c + d*x]])*(Cos[(c + d*x)/2] -
Sin[(c + d*x)/2]))/(7*d*Csc[c + d*x]^(1/3)*(Cos[(c + d*x)/2] + Sin[(c + d*
x)/2]))
```

3.24.3 Rubi [A] (warning: unable to verify)

Time = 0.48 (sec) , antiderivative size = 510, normalized size of antiderivative = 0.99, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.280$, Rules used = {3042, 4293, 60, 73, 832, 759, 2416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a\csc(c+dx)+a}\,dx$$

$$\Big\downarrow \; 3042$$

$$\int \csc(c+dx)^{5/3}\sqrt{a\csc(c+dx)+a}dx$$

$$\Big\downarrow \; 4293$$

$$\frac{a^2\cot(c+dx)\int \frac{\csc^{\frac{2}{3}}(c+dx)}{\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \; 60$$

$$\frac{a^2\cot(c+dx)\left(\frac{4}{7}\int \frac{1}{\sqrt[3]{\csc(c+dx)}\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)-\frac{6\csc^{\frac{2}{3}}(c+dx)\sqrt{a-a\csc(c+dx)}}{7a}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \; 73$$

$$\frac{a^2\cot(c+dx)\left(\frac{12}{7}\int \frac{\sqrt[3]{\csc(c+dx)}}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\frac{6\csc^{\frac{2}{3}}(c+dx)\sqrt{a-a\csc(c+dx)}}{7a}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \; 832$$

$$\frac{a^2\cot(c+dx)\left(\frac{12}{7}\left(\left(1-\sqrt{3}\right)\int \frac{1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\int \frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}\right)-\frac{6\csc^{\frac{2}{3}}(\cdots}{\cdots}\right.}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \; 759$$

$$\frac{a^2\cot(c+dx)\left(\frac{12}{7}\left(-\int \frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\frac{2\left(1-\sqrt{3}\right)\sqrt{2+\sqrt{3}}\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\cdots}}{\left(-\sqrt[3]{\csc(c+\cdots}\right.}}}{\sqrt[4]{3}\sqrt{\frac{1-\sqrt[3]{\cdots}}{\left(-\sqrt[3]{\csc(\cdots}\right.}}}\right.\right.}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c\cdots}}$$

$$\Big\downarrow \; 2416$$

3.24. $\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$

$$\frac{a^2\cot(c+dx)\left(\frac{12}{7}\left(-\frac{2\left(1-\sqrt{3}\right)\sqrt{2+\sqrt{3}}\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\,\text{EllipticF}\left(\arcsin\left(\frac{-\sqrt[3]{\csc(c+d}}{-\sqrt[3]{\csc(c+d}}\right.\right.}{\sqrt[4]{3}\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\sqrt{a-a\csc(c+dx)}}\right.\right.}{}$$

input
```
Int[Csc[c + d*x]^(5/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

output
```
(a^2*Cot[c + d*x]*((-6*Csc[c + d*x]^(2/3)*Sqrt[a - a*Csc[c + d*x]])/(7*a)
+ (12*((2*Sqrt[a - a*Csc[c + d*x]])/(a*(1 + Sqrt[3] - Csc[c + d*x]^(1/3)))
 - (3^(1/4)*Sqrt[2 - Sqrt[3]]*(1 - Csc[c + d*x]^(1/3))*Sqrt[(1 + Csc[c + d
*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*Elli
pticE[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x
]^(1/3))], -7 - 4*Sqrt[3]])/(Sqrt[(1 - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] -
Csc[c + d*x]^(1/3))^2]*Sqrt[a - a*Csc[c + d*x]]) - (2*(1 - Sqrt[3])*Sqrt[2
 + Sqrt[3]]*(1 - Csc[c + d*x]^(1/3))*Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[c
+ d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*EllipticF[ArcSin[(1 -
Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))], -7 - 4*
Sqrt[3]])/(3^(1/4)*Sqrt[(1 - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*
x]^(1/3))^2]*Sqrt[a - a*Csc[c + d*x]])))/7))/(d*Sqrt[a - a*Csc[c + d*x]]*S
qrt[a + a*Csc[c + d*x]])
```

3.24.3.1 Defintions of rubi rules used

rule 60
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Simp[n*((b*c - a*d)/(
b*(m + n + 1)))   Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
 c, d}, x] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ[m, 0] && ( !Integer
Q[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n + 2, 0] && IntLinear
Q[a, b, c, d, m, n, x]
```

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 759
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[2*Sqrt[2 + Sqrt[3]]*(s + r*x)*(Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]/(3^(1/4)*r*Sqrt[a + b*x^3]*Sqrt[s*
((s + r*x)/((1 + Sqrt[3])*s + r*x)^2)]))*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]], x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

rule 832
```
Int[(x_)/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]
], s = Denom[Rt[b/a, 3]]}, Simp[(-(1 - Sqrt[3]))*(s/r)   Int[1/Sqrt[a + b*x
^3], x], x] + Simp[1/r   Int[((1 - Sqrt[3])*s + r*x)/Sqrt[a + b*x^3], x], x
]] /; FreeQ[{a, b}, x] && PosQ[a]
```

rule 2416
```
Int[((c_) + (d_.)*(x_))/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = N
umer[Simplify[(1 - Sqrt[3])*(d/c)]], s = Denom[Simplify[(1 - Sqrt[3])*(d/c)
]]}, Simp[2*d*s^3*(Sqrt[a + b*x^3]/(a*r^2*((1 + Sqrt[3])*s + r*x))), x] - S
imp[3^(1/4)*Sqrt[2 - Sqrt[3]]*d*s*(s + r*x)*(Sqrt[(s^2 - r*s*x + r^2*x^2)/(
(1 + Sqrt[3])*s + r*x)^2]/(r^2*Sqrt[a + b*x^3]*Sqrt[s*((s + r*x)/((1 + Sqrt
[3])*s + r*x)^2)]))*EllipticE[ArcSin[((1 - Sqrt[3])*s + r*x)/((1 + Sqrt[3])
*s + r*x)], -7 - 4*Sqrt[3]], x]] /; FreeQ[{a, b, c, d}, x] && PosQ[a] && Eq
Q[b*c^3 - 2*(5 - 3*Sqrt[3])*a*d^3, 0]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4293
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x]]
*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
 Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

3.24.4 Maple [F]

$$\int \csc(dx+c)^{\frac{5}{3}} \sqrt{a+a\csc(dx+c)}dx$$

input
```
int(csc(d*x+c)^(5/3)*(a+a*csc(d*x+c))^(1/2),x)
```

output
```
int(csc(d*x+c)^(5/3)*(a+a*csc(d*x+c))^(1/2),x)
```

3.24.5 Fricas [F]

$$\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx=\int \sqrt{a\csc(dx+c)+a}\csc(dx+c)^{\frac{5}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(5/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="fricas")
```

output
```
integral(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(5/3), x)
```

3.24.6 Sympy [F(-1)]

Timed out.

$$\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx=\text{Timed out}$$

input
```
integrate(csc(d*x+c)**(5/3)*(a+a*csc(d*x+c))**(1/2),x)
```

output
```
Timed out
```

3.24.7 Maxima [F]

$$\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx=\int \sqrt{a\csc(dx+c)+a}\csc(dx+c)^{\frac{5}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(5/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="maxima")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(5/3), x)
```

3.24.8 Giac [F]

$$\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a\csc(dx+c)+a}\csc(dx+c)^{\frac{5}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(5/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="giac")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(5/3), x)
```

3.24.9 Mupad [F(-1)]

Timed out.

$$\int \csc^{\frac{5}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a+\frac{a}{\sin(c+d\,x)}}\left(\frac{1}{\sin(c+d\,x)}\right)^{5/3}dx$$

input
```
int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(5/3),x)
```

output
```
int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(5/3), x)
```

3.25 $\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$

3.25.1 Optimal result

Integrand size = 25, antiderivative size = 470

$$\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \frac{6a\cot(c+dx)}{d\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)\sqrt{a+a\csc(c+dx)}}$$

$$-\frac{3\sqrt[4]{3}\sqrt{2-\sqrt{3}}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\csc^{\frac{2}{3}}(c+dx)}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}E\left(\arcsin\left(\frac{1-\sqrt{3}-\sqrt[3]{\csc}}{1+\sqrt{3}-\sqrt[3]{\csc}}\right.\right.}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}(a-a\csc(c+dx))\sqrt{a+a\csc(c+dx)}}$$

$$+\frac{2\sqrt{2}3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\csc^{\frac{2}{3}}(c+dx)}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}\operatorname{EllipticF}\left(\arcsin\left(\frac{1-\sqrt{3}-\sqrt[3]{c}}{1+\sqrt{3}-\sqrt[3]{c}}\right.\right.}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}(a-a\csc(c+dx))\sqrt{a+a\csc(c+dx)}}$$

output
```
6*a*cot(d*x+c)/d/(1-csc(d*x+c)^(1/3)+3^(1/2))/(a+a*csc(d*x+c))^(1/2)+2*3^(
3/4)*a^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(
1/2))/(1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*2^(1/2)*((1+csc(d*x+c)^(
1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/(a-a*csc(d*
x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c)^(1/3)+3^(
1/2))^2)^(1/2)-3*3^(1/4)*a^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticE((1-
csc(d*x+c)^(1/3)-3^(1/2))/(1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2
*6^(1/2)-1/2*2^(1/2))*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)
^(1/3)+3^(1/2))^2)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc
(d*x+c)^(1/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)
```

3.25.2 Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 4 in optimal.

Time = 15.58 (sec) , antiderivative size = 109, normalized size of antiderivative = 0.23

$$\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$$
$$=\frac{2\sqrt{a(1+\csc(c+dx))}\left(-3+2\sqrt[3]{\csc(c+dx)}\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{4}{3},\frac{3}{2},1-\csc(c+dx)\right)\right)\left(\cos\left(\frac{1}{2}(c+\right.\right.}{d\sqrt[3]{\csc(c+dx)}\left(\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)\right)}$$

input
```
Integrate[Csc[c + d*x]^(2/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

output
```
(2*Sqrt[a*(1 + Csc[c + d*x])]*(-3 + 2*Csc[c + d*x]^(1/3)*Hypergeometric2F1
[1/2, 4/3, 3/2, 1 - Csc[c + d*x]])*(Cos[(c + d*x)/2] - Sin[(c + d*x)/2]))/
(d*Csc[c + d*x]^(1/3)*(Cos[(c + d*x)/2] + Sin[(c + d*x)/2]))
```

3.25.3 Rubi [A] (warning: unable to verify)

Time = 0.48 (sec) , antiderivative size = 474, normalized size of antiderivative = 1.01, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.240, Rules used = {3042, 4293, 73, 832, 759, 2416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a\csc(c+dx)+a}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \csc(c+dx)^{2/3}\sqrt{a\csc(c+dx)+a}dx$$

$$\Big\downarrow \text{4293}$$

$$\frac{a^2\cot(c+dx)\int \frac{1}{\sqrt[3]{\csc(c+dx)}\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \text{73}$$

$$\frac{3a^2\cot(c+dx)\int \frac{\sqrt[3]{\csc(c+dx)}}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \text{832}$$

$$\frac{3a^2\cot(c+dx)\left(\left(1-\sqrt{3}\right)\int \frac{1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\int \frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \text{759}$$

$$\frac{3a^2\cot(c+dx)\left(-\int \frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\frac{2\left(1-\sqrt{3}\right)\sqrt{2+\sqrt{3}}\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(\ldots}}{\left(-\sqrt[3]{\csc(c+d\ldots}}}}{\sqrt[4]{3}\sqrt{\frac{1-\sqrt[3]{\csc(\ldots}}{\left(-\sqrt[3]{\csc(c+\ldots}}}}\ldots}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}\ldots}$$

$$\Big\downarrow \text{2416}$$

$$\frac{3a^2\cot(c+dx)\left(-\frac{2\left(1-\sqrt{3}\right)\sqrt{2+\sqrt{3}}\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\text{EllipticF}\left(\arcsin\left(\frac{-\sqrt[3]{\csc(c+dx)}-\ldots}{-\sqrt[3]{\csc(c+dx)}+\ldots}\right.\right.}{\sqrt[4]{3}\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\sqrt{a-a\csc(c+dx)}}\ldots}{d\sqrt{\ldots}}$$

input
```
Int[Csc[c + d*x]^(2/3)*Sqrt[a + a*Csc[c + d*x]],x]
```

3.25. $\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$

output
```
(3*a^2*Cot[c + d*x]*((2*Sqrt[a - a*Csc[c + d*x]])/(a*(1 + Sqrt[3] - Csc[c
+ d*x]^(1/3))) - (3^(1/4)*Sqrt[2 - Sqrt[3]]*(1 - Csc[c + d*x]^(1/3))*Sqrt[
(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]^
(1/3))^2]*EllipticE[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3]
 - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(Sqrt[(1 - Csc[c + d*x]^(1/3))/(
1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*Sqrt[a - a*Csc[c + d*x]]) - (2*(1 - S
qrt[3])*Sqrt[2 + Sqrt[3]]*(1 - Csc[c + d*x]^(1/3))*Sqrt[(1 + Csc[c + d*x]^
(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*Elliptic
F[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1
/3))], -7 - 4*Sqrt[3]])/(3^(1/4)*Sqrt[(1 - Csc[c + d*x]^(1/3))/(1 + Sqrt[3
] - Csc[c + d*x]^(1/3))^2]*Sqrt[a - a*Csc[c + d*x]])))/(d*Sqrt[a - a*Csc[c
 + d*x]]*Sqrt[a + a*Csc[c + d*x]])
```

3.25.3.1 Defintions of rubi rules used

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 759
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[2*Sqrt[2 + Sqrt[3]]*(s + r*x)*(Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]/(3^(1/4)*r*Sqrt[a + b*x^3]*Sqrt[s*
((s + r*x)/((1 + Sqrt[3])*s + r*x)^2)]))*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]], x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

rule 832
```
Int[(x_)/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]
], s = Denom[Rt[b/a, 3]]}, Simp[(-(1 - Sqrt[3]))*(s/r)   Int[1/Sqrt[a + b*x
^3], x], x] + Simp[1/r   Int[((1 - Sqrt[3])*s + r*x)/Sqrt[a + b*x^3], x], x
]] /; FreeQ[{a, b}, x] && PosQ[a]
```

rule 2416
```
Int[((c_) + (d_.)*(x_))/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = N
umer[Simplify[(1 - Sqrt[3])*(d/c)]], s = Denom[Simplify[(1 - Sqrt[3])*(d/c)
]]}, Simp[2*d*s^3*(Sqrt[a + b*x^3]/(a*r^2*((1 + Sqrt[3])*s + r*x))), x] - S
imp[3^(1/4)*Sqrt[2 - Sqrt[3]]*d*s*(s + r*x)*(Sqrt[(s^2 - r*s*x + r^2*x^2)/(
(1 + Sqrt[3])*s + r*x)^2]/(r^2*Sqrt[a + b*x^3]*Sqrt[s*((s + r*x)/((1 + Sqrt
[3])*s + r*x)^2)]))*EllipticE[ArcSin[((1 - Sqrt[3])*s + r*x)/((1 + Sqrt[3])
*s + r*x)], -7 - 4*Sqrt[3]], x]] /; FreeQ[{a, b, c, d}, x] && PosQ[a] && Eq
Q[b*c^3 - 2*(5 - 3*Sqrt[3])*a*d^3, 0]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4293
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x]]
*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
 Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

3.25.4 Maple [F]

$$\int \csc\left(dx+c\right)^{\frac{2}{3}} \sqrt{a+a\csc\left(dx+c\right)}dx$$

input
```
int(csc(d*x+c)^(2/3)*(a+a*csc(d*x+c))^(1/2),x)
```

output
```
int(csc(d*x+c)^(2/3)*(a+a*csc(d*x+c))^(1/2),x)
```

3.25.5 Fricas [F]

$$\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a\csc\left(dx+c\right)+a}\csc\left(dx+c\right)^{\frac{2}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(2/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="fricas")
```

output
```
integral(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(2/3), x)
```

3.25.6 Sympy [F]

$$\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx=\int \sqrt{a\left(\csc\left(c+dx\right)+1\right)}\csc^{\frac{2}{3}}\left(c+dx\right)dx$$

input
```
integrate(csc(d*x+c)**(2/3)*(a+a*csc(d*x+c))**(1/2),x)
```

output
```
Integral(sqrt(a*(csc(c + d*x) + 1))*csc(c + d*x)**(2/3), x)
```

3.25.7 Maxima [F]

$$\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx=\int \sqrt{a\csc\left(dx+c\right)+a}\csc\left(dx+c\right)^{\frac{2}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(2/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="maxima")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(2/3), x)
```

3.25.8 Giac [F]

$$\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx=\int \sqrt{a\csc\left(dx+c\right)+a}\csc\left(dx+c\right)^{\frac{2}{3}}\,dx$$

input
```
integrate(csc(d*x+c)^(2/3)*(a+a*csc(d*x+c))^(1/2),x, algorithm="giac")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^(2/3), x)
```

3.25.9 Mupad [F(-1)]

Timed out.

$$\int \csc^{\frac{2}{3}}(c+dx)\sqrt{a+a\csc(c+dx)}\,dx=\int \sqrt{a+\frac{a}{\sin\left(c+d\,x\right)}}\,{\left(\frac{1}{\sin\left(c+d\,x\right)}\right)}^{2/3}\,dx$$

input
```
int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(2/3),x)
```

output
```
int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^(2/3), x)
```

3.26 $\int \frac{\sqrt{a+a\csc(c+dx)}}{\sqrt[3]{\csc(c+dx)}}\,dx$

3.26.1 Optimal result

Integrand size = 25, antiderivative size = 508

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\sqrt[3]{\csc(c+dx)}}\,dx = -\frac{3a\cot(c+dx)}{d\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)\sqrt{a+a\csc(c+dx)}}-\frac{3a\cos(c+dx)\csc^{\frac{2}{3}}(c+dx)}{d\sqrt{a+a\csc(c+dx)}}$$
$$+\frac{3\sqrt[4]{3}\sqrt{2-\sqrt{3}}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\csc^{\frac{2}{3}}(c+dx)}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}E\left(\arcsin\left(\frac{1-\sqrt{3}-\sqrt[3]{\csc}}{1+\sqrt{3}-\sqrt[3]{\csc}}\right.\right.}{2d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}(a-a\csc(c+dx))\sqrt{a+a\csc(c+dx)}}$$
$$-\frac{\sqrt{2}3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\csc^{\frac{2}{3}}(c+dx)}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}\operatorname{EllipticF}\left(\arcsin\left(\frac{1-\sqrt{3}-\sqrt[3]{\text{cs}}}{1+\sqrt{3}-\sqrt[3]{\text{cs}}}\right.\right.}{d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}(a-a\csc(c+dx))\sqrt{a+a\csc(c+dx)}}$$

output
```
-3*a*cos(d*x+c)*csc(d*x+c)^(2/3)/d/(a+a*csc(d*x+c))^(1/2)-3*a*cot(d*x+c)/d
/(1-csc(d*x+c)^(1/3)+3^(1/2))/(a+a*csc(d*x+c))^(1/2)-3^(3/4)*a^2*cot(d*x+c
)*(1-csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(1/2))/(1-csc(d*x+c
)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*2^(1/2)*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2
/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x
+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)+3/2
*3^(1/4)*a^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticE((1-csc(d*x+c)^(1/3)
-3^(1/2))/(1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2)-1/2*2^(
1/2))*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^
2)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(
1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)
```

3.26.2 Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 4 in optimal.

Time = 15.30 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.09

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\sqrt[3]{\csc(c+dx)}}\,dx = -\frac{2a\cot(c+dx)\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{4}{3},\frac{3}{2},1-\csc(c+dx)\right)}{d\sqrt{a(1+\csc(c+dx))}}$$

input
```
Integrate[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(1/3),x]
```

output
```
(-2*a*Cot[c + d*x]*Hypergeometric2F1[1/2, 4/3, 3/2, 1 - Csc[c + d*x]])/(d*
Sqrt[a*(1 + Csc[c + d*x])])
```

3.26.3 Rubi [A] (warning: unable to verify)

Time = 0.48 (sec) , antiderivative size = 508, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.280$, Rules used = {3042, 4293, 61, 73, 832, 759, 2416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{a\csc(c+dx)+a}}{\sqrt[3]{\csc(c+dx)}}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\sqrt{a\csc(c+dx)+a}}{\sqrt[3]{\csc(c+dx)}}dx$$

$\big\downarrow$ 4293

$$\frac{a^2\cot(c+dx)\int \frac{1}{\csc^{\frac{4}{3}}(c+dx)\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$\big\downarrow$ 61

$$\frac{a^2\cot(c+dx)\left(-\frac{1}{2}\int \frac{1}{\sqrt[3]{\csc(c+dx)}\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)-\frac{3\sqrt{a-a\csc(c+dx)}}{a\sqrt[3]{\csc(c+dx)}}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$\big\downarrow$ 73

$$\frac{a^2\cot(c+dx)\left(-\frac{3}{2}\int \frac{\sqrt[3]{\csc(c+dx)}}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\frac{3\sqrt{a-a\csc(c+dx)}}{a\sqrt[3]{\csc(c+dx)}}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$\big\downarrow$ 832

$$\frac{a^2\cot(c+dx)\left(-\frac{3}{2}\left(\left(1-\sqrt{3}\right)\int \frac{1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\int \frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}\right)-\frac{3\sqrt{a-}}{a\sqrt[3]{c}}\right.}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$\big\downarrow$ 759

$$\frac{a^2\cot(c+dx)\left(-\frac{3}{2}\left(-\int \frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\frac{2\left(1-\sqrt{3}\right)\sqrt{2+\sqrt{3}}\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{}}{\left(-\sqrt[3]{\csc(c}\right.}}}{\sqrt[4]{3}\sqrt{\frac{1-\sqrt[3]{}}{\left(-\sqrt[3]{\csc(}\right.}}}\right.\right.}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx}}$$

$\big\downarrow$ 2416

$$\frac{a^2\cot(c+dx)\left(-\frac{3}{2}\left(-\frac{2\left(1-\sqrt{3}\right)\sqrt{2+\sqrt{3}}\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\text{EllipticF}\left(\arcsin\left(\frac{-\sqrt[3]{\csc(c+}}{-\sqrt[3]{\csc(c+}}\right.\right.}{\sqrt[4]{3}\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\sqrt{a-a\csc(c+dx)}}\right.\right.}{}$$

3.26. $\int \frac{\sqrt{a+a\csc(c+dx)}}{\sqrt[3]{\csc(c+dx)}}dx$

input
```
Int[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(1/3),x]
```

output
```
(a^2*Cot[c + d*x]*((-3*Sqrt[a - a*Csc[c + d*x]])/(a*Csc[c + d*x]^(1/3)) -
(3*((2*Sqrt[a - a*Csc[c + d*x]])/(a*(1 + Sqrt[3] - Csc[c + d*x]^(1/3))) -
(3^(1/4)*Sqrt[2 - Sqrt[3]]*(1 - Csc[c + d*x]^(1/3))*Sqrt[(1 + Csc[c + d*x]
^(1/3) + Csc[c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*Ellipti
cE[ArcSin[(1 - Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(
1/3))], -7 - 4*Sqrt[3]])/(Sqrt[(1 - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc
[c + d*x]^(1/3))^2]*Sqrt[a - a*Csc[c + d*x]]) - (2*(1 - Sqrt[3])*Sqrt[2 +
Sqrt[3]]*(1 - Csc[c + d*x]^(1/3))*Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[c + d
*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*EllipticF[ArcSin[(1 - Sqr
t[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))], -7 - 4*Sqr
t[3]])/(3^(1/4)*Sqrt[(1 - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^
(1/3))^2]*Sqrt[a - a*Csc[c + d*x]])))/2))/(d*Sqrt[a - a*Csc[c + d*x]]*Sqrt
[a + a*Csc[c + d*x]])
```

3.26.3.1 Defintions of rubi rules used

rule 61
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^(n + 1)/((b*c - a*d)*(m + 1))), x] - Simp[d*((
m + n + 2)/((b*c - a*d)*(m + 1)))   Int[(a + b*x)^(m + 1)*(c + d*x)^n, x],
x] /; FreeQ[{a, b, c, d, n}, x] && LtQ[m, -1] &&  !(LtQ[n, -1] && (EqQ[a, 0
] || (NeQ[c, 0] && LtQ[m - n, 0] && IntegerQ[n]))) && IntLinearQ[a, b, c, d
, m, n, x]
```

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 759
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[2*Sqrt[2 + Sqrt[3]]*(s + r*x)*(Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]/(3^(1/4)*r*Sqrt[a + b*x^3]*Sqrt[s*
((s + r*x)/((1 + Sqrt[3])*s + r*x)^2)]))*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]], x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

rule 832
```
Int[(x_)/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]
], s = Denom[Rt[b/a, 3]]}, Simp[(-(1 - Sqrt[3]))*(s/r)   Int[1/Sqrt[a + b*x
^3], x], x] + Simp[1/r   Int[((1 - Sqrt[3])*s + r*x)/Sqrt[a + b*x^3], x], x
]] /; FreeQ[{a, b}, x] && PosQ[a]
```

rule 2416
```
Int[((c_) + (d_.)*(x_))/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = N
umer[Simplify[(1 - Sqrt[3])*(d/c)]], s = Denom[Simplify[(1 - Sqrt[3])*(d/c)
]]}, Simp[2*d*s^3*(Sqrt[a + b*x^3]/(a*r^2*((1 + Sqrt[3])*s + r*x))), x] - S
imp[3^(1/4)*Sqrt[2 - Sqrt[3]]*d*s*(s + r*x)*(Sqrt[(s^2 - r*s*x + r^2*x^2)/(
(1 + Sqrt[3])*s + r*x)^2]/(r^2*Sqrt[a + b*x^3]*Sqrt[s*((s + r*x)/((1 + Sqrt
[3])*s + r*x)^2)]))*EllipticE[ArcSin[((1 - Sqrt[3])*s + r*x)/((1 + Sqrt[3])
*s + r*x)], -7 - 4*Sqrt[3]], x]] /; FreeQ[{a, b, c, d}, x] && PosQ[a] && Eq
Q[b*c^3 - 2*(5 - 3*Sqrt[3])*a*d^3, 0]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4293
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x]]
*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
 Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

3.26.4 Maple [F]

$$\int \frac{\sqrt{a+a\csc\left(dx+c\right)}}{\csc\left(dx+c\right)^{\frac{1}{3}}}dx$$

input
```
int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(1/3),x)
```

output
```
int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(1/3),x)
```

3.26.5 Fricas [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\sqrt[3]{\csc(c+dx)}}\,dx = \int \frac{\sqrt{a\csc\left(dx+c\right)+a}}{\csc\left(dx+c\right)^{\frac{1}{3}}}\,dx$$

input
```
integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(1/3),x, algorithm="fricas")
```

output
```
integral(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(1/3), x)
```

3.26.6 Sympy [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\sqrt[3]{\csc(c+dx)}}\,dx = \int \frac{\sqrt{a\left(\csc\left(c+dx\right)+1\right)}}{\sqrt[3]{\csc\left(c+dx\right)}}\,dx$$

input
```
integrate((a+a*csc(d*x+c))**(1/2)/csc(d*x+c)**(1/3),x)
```

output
```
Integral(sqrt(a*(csc(c + d*x) + 1))/csc(c + d*x)**(1/3), x)
```

3.26.7 Maxima [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\sqrt[3]{\csc(c+dx)}}\,dx = \int \frac{\sqrt{a\csc\left(dx+c\right)+a}}{\csc\left(dx+c\right)^{\frac{1}{3}}}\,dx$$

input
```
integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(1/3),x, algorithm="maxima")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(1/3), x)
```

3.26.8 Giac [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\sqrt[3]{\csc(c+dx)}}\,dx=\int \frac{\sqrt{a\csc(dx+c)+a}}{\csc(dx+c)^{\frac{1}{3}}}\,dx$$

input
```
integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(1/3),x, algorithm="giac")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(1/3), x)
```

3.26.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\sqrt[3]{\csc(c+dx)}}\,dx=\int \frac{\sqrt{a+\frac{a}{\sin(c+d\,x)}}}{\left(\frac{1}{\sin(c+d\,x)}\right)^{1/3}}\,dx$$

input
```
int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(1/3),x)
```

output
```
int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(1/3), x)
```

3.27 $\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx$

3.27.1 Optimal result

Integrand size = 25, antiderivative size = 552

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx = -\frac{15a\cot(c+dx)}{8d\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)\sqrt{a+a\csc(c+dx)}} - \frac{3a\cos(c+dx)}{4d\sqrt[3]{\csc(c+dx)}\sqrt{a+a\csc(c+dx)}} - \frac{15a\cos(c+dx)\csc^{\frac{2}{3}}(c+dx)}{8d\sqrt{a+a\csc(c+dx)}} + \frac{15\sqrt[4]{3}\sqrt{2-\sqrt{3}}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\csc^{\frac{2}{3}}(c+dx)}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}E\left(\arcsin\left(\frac{1-\sqrt{3}-\sqrt[3]{\csc}}{1+\sqrt{3}-\sqrt[3]{\csc}}\right.\right.}{16d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}(a-a\csc(c+dx))\sqrt{a+a\csc(c+dx)}} - \frac{5\,3^{3/4}a^2\cot(c+dx)\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{1+\sqrt[3]{\csc(c+dx)}+\csc^{\frac{2}{3}}(c+dx)}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}\operatorname{EllipticF}\left(\arcsin\left(\frac{1-\sqrt{3}-\sqrt[3]{\csc}}{1+\sqrt{3}-\sqrt[3]{\csc}}\right.\right.}{4\sqrt{2}d\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(1+\sqrt{3}-\sqrt[3]{\csc(c+dx)}\right)^2}}(a-a\csc(c+dx))\sqrt{a+a\csc(c+dx)}}$$

output
```
-3/4*a*cos(d*x+c)/d/csc(d*x+c)^(1/3)/(a+a*csc(d*x+c))^(1/2)-15/8*a*cos(d*x
+c)*csc(d*x+c)^(2/3)/d/(a+a*csc(d*x+c))^(1/2)-15/8*a*cot(d*x+c)/d/(1-csc(d
*x+c)^(1/3)+3^(1/2))/(a+a*csc(d*x+c))^(1/2)-5/8*3^(3/4)*a^2*cot(d*x+c)*(1-
csc(d*x+c)^(1/3))*EllipticF((1-csc(d*x+c)^(1/3)-3^(1/2))/(1-csc(d*x+c)^(1/
3)+3^(1/2)),I*3^(1/2)+2*I)*2^(1/2)*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/
(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^
(1/2)/((1-csc(d*x+c)^(1/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)+15/16*3^
(1/4)*a^2*cot(d*x+c)*(1-csc(d*x+c)^(1/3))*EllipticE((1-csc(d*x+c)^(1/3)-3^
(1/2))/(1-csc(d*x+c)^(1/3)+3^(1/2)),I*3^(1/2)+2*I)*(1/2*6^(1/2)-1/2*2^(1/2
))*((1+csc(d*x+c)^(1/3)+csc(d*x+c)^(2/3))/(1-csc(d*x+c)^(1/3)+3^(1/2))^2)^
(1/2)/d/(a-a*csc(d*x+c))/(a+a*csc(d*x+c))^(1/2)/((1-csc(d*x+c)^(1/3))/(1-c
sc(d*x+c)^(1/3)+3^(1/2))^2)^(1/2)
```

3.27.2 Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 4 in optimal.

Time = 15.88 (sec) , antiderivative size = 72, normalized size of antiderivative = 0.13

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx = -\frac{a\cos(c+dx)\left(3+5\csc^{\frac{4}{3}}(c+dx)\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{4}{3},\frac{3}{2},1-\csc(c+dx)\right)\right)}{4d\sqrt[3]{\csc(c+dx)}\sqrt{a(1+\csc(c+dx))}}$$

input
```
Integrate[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(4/3),x]
```

output
```
-1/4*(a*Cos[c + d*x]*(3 + 5*Csc[c + d*x]^(4/3)*Hypergeometric2F1[1/2, 4/3,
 3/2, 1 - Csc[c + d*x]]))/(d*Csc[c + d*x]^(1/3)*Sqrt[a*(1 + Csc[c + d*x])]
)
```

3.27.3 Rubi [A] (warning: unable to verify)

Time = 0.50 (sec) , antiderivative size = 545, normalized size of antiderivative = 0.99, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.320, Rules used = {3042, 4293, 61, 61, 73, 832, 759, 2416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{a\csc(c+dx)+a}}{\csc^{\frac{4}{3}}(c+dx)}\,dx$$

$$\Big\downarrow \; 3042$$

$$\int \frac{\sqrt{a\csc(c+dx)+a}}{\csc(c+dx)^{4/3}}dx$$

$$\Big\downarrow \; 4293$$

$$\frac{a^2\cot(c+dx)\int \frac{1}{\csc^{\frac{7}{3}}(c+dx)\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \; 61$$

$$\frac{a^2\cot(c+dx)\left(\frac{5}{8}\int \frac{1}{\csc^{\frac{4}{3}}(c+dx)\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)-\frac{3\sqrt{a-a\csc(c+dx)}}{4a\csc^{\frac{4}{3}}(c+dx)}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \; 61$$

$$\frac{a^2\cot(c+dx)\left(\frac{5}{8}\left(-\frac{1}{2}\int \frac{1}{\sqrt[3]{\csc(c+dx)}\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)-\frac{3\sqrt{a-a\csc(c+dx)}}{a\sqrt[3]{\csc(c+dx)}}\right)-\frac{3\sqrt{a-a\csc(c+dx)}}{4a\csc^{\frac{4}{3}}(c+dx)}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \; 73$$

$$\frac{a^2\cot(c+dx)\left(\frac{5}{8}\left(-\frac{3}{2}\int \frac{\sqrt[3]{\csc(c+dx)}}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\frac{3\sqrt{a-a\csc(c+dx)}}{a\sqrt[3]{\csc(c+dx)}}\right)-\frac{3\sqrt{a-a\csc(c+dx)}}{4a\csc^{\frac{4}{3}}(c+dx)}\right)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \; 832$$

$$\frac{a^2\cot(c+dx)\left(\frac{5}{8}\left(-\frac{3}{2}\left(\left(1-\sqrt{3}\right)\int \frac{1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\int \frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}\right)-\frac{3\cdots}{a\cdots}\right.\right.}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \; 759$$

$$\frac{a^2\cot(c+dx)\left(\frac{5}{8}\left(-\frac{3}{2}\left(-\int \frac{-\sqrt[3]{\csc(c+dx)}-\sqrt{3}+1}{\sqrt{a-a\csc(c+dx)}}d\sqrt[3]{\csc(c+dx)}-\frac{2\left(1-\sqrt{3}\right)\sqrt{2+\sqrt{3}}\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)\cdots}{\left(-\sqrt[3]{\csc\cdots}\right.}}}{\sqrt[4]{3}\sqrt{\frac{1-\cdots}{\left(-\sqrt[3]{\cdots}\right.}}}\right.\right.\right.}{d\sqrt{a-a\csc(c+dx)}\sqrt{\cdots}}$$

3.27. $\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx$

$$\Big\downarrow \text{2416}$$

$$\frac{a^2\cot(c+dx)\left(\frac{5}{8}\left(-\frac{3}{2}\left(-\frac{2\left(1-\sqrt{3}\right)\sqrt{2+\sqrt{3}}\left(1-\sqrt[3]{\csc(c+dx)}\right)\sqrt{\frac{\csc^{\frac{2}{3}}(c+dx)+\sqrt[3]{\csc(c+dx)}+1}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\text{EllipticF}\left(\arcsin\left(\frac{-\sqrt[3]{\csc(c}}{-\sqrt[3]{\csc(c}}\right.\right.}{\sqrt[4]{3}\sqrt{\frac{1-\sqrt[3]{\csc(c+dx)}}{\left(-\sqrt[3]{\csc(c+dx)}+\sqrt{3}+1\right)^2}}\sqrt{a-a\csc(c+dx)}}\right.\right.\right.}{}$$

input
```
Int[Sqrt[a + a*Csc[c + d*x]]/Csc[c + d*x]^(4/3),x]
```

output
```
(a^2*Cot[c + d*x]*((-3*Sqrt[a - a*Csc[c + d*x]])/(4*a*Csc[c + d*x]^(4/3))
+ (5*((-3*Sqrt[a - a*Csc[c + d*x]])/(a*Csc[c + d*x]^(1/3)) - (3*((2*Sqrt[a
 - a*Csc[c + d*x]])/(a*(1 + Sqrt[3] - Csc[c + d*x]^(1/3))) - (3^(1/4)*Sqrt
[2 - Sqrt[3]]*(1 - Csc[c + d*x]^(1/3))*Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[
c + d*x]^(2/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*EllipticE[ArcSin[(1
- Sqrt[3] - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))], -7 -
4*Sqrt[3]])/(Sqrt[(1 - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/
3))^2]*Sqrt[a - a*Csc[c + d*x]]) - (2*(1 - Sqrt[3])*Sqrt[2 + Sqrt[3]]*(1 -
 Csc[c + d*x]^(1/3))*Sqrt[(1 + Csc[c + d*x]^(1/3) + Csc[c + d*x]^(2/3))/(1
 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*EllipticF[ArcSin[(1 - Sqrt[3] - Csc[c
+ d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))], -7 - 4*Sqrt[3]])/(3^(1/
4)*Sqrt[(1 - Csc[c + d*x]^(1/3))/(1 + Sqrt[3] - Csc[c + d*x]^(1/3))^2]*Sqr
t[a - a*Csc[c + d*x]])))/2))/8))/(d*Sqrt[a - a*Csc[c + d*x]]*Sqrt[a + a*Cs
c[c + d*x]])
```

3.27.3.1 Defintions of rubi rules used

rule 61
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^(n + 1)/((b*c - a*d)*(m + 1))), x] - Simp[d*((
m + n + 2)/((b*c - a*d)*(m + 1)))   Int[(a + b*x)^(m + 1)*(c + d*x)^n, x],
x] /; FreeQ[{a, b, c, d, n}, x] && LtQ[m, -1] &&  !(LtQ[n, -1] && (EqQ[a, 0
] || (NeQ[c, 0] && LtQ[m - n, 0] && IntegerQ[n]))) && IntLinearQ[a, b, c, d
, m, n, x]
```

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 759
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]],
s = Denom[Rt[b/a, 3]]}, Simp[2*Sqrt[2 + Sqrt[3]]*(s + r*x)*(Sqrt[(s^2 - r*s
*x + r^2*x^2)/((1 + Sqrt[3])*s + r*x)^2]/(3^(1/4)*r*Sqrt[a + b*x^3]*Sqrt[s*
((s + r*x)/((1 + Sqrt[3])*s + r*x)^2)]))*EllipticF[ArcSin[((1 - Sqrt[3])*s
+ r*x)/((1 + Sqrt[3])*s + r*x)], -7 - 4*Sqrt[3]], x]] /; FreeQ[{a, b}, x] &
& PosQ[a]
```

rule 832
```
Int[(x_)/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = Numer[Rt[b/a, 3]
], s = Denom[Rt[b/a, 3]]}, Simp[(-(1 - Sqrt[3]))*(s/r)   Int[1/Sqrt[a + b*x
^3], x], x] + Simp[1/r   Int[((1 - Sqrt[3])*s + r*x)/Sqrt[a + b*x^3], x], x
]] /; FreeQ[{a, b}, x] && PosQ[a]
```

rule 2416
```
Int[((c_) + (d_.)*(x_))/Sqrt[(a_) + (b_.)*(x_)^3], x_Symbol] :> With[{r = N
umer[Simplify[(1 - Sqrt[3])*(d/c)]], s = Denom[Simplify[(1 - Sqrt[3])*(d/c)
]]}, Simp[2*d*s^3*(Sqrt[a + b*x^3]/(a*r^2*((1 + Sqrt[3])*s + r*x))), x] - S
imp[3^(1/4)*Sqrt[2 - Sqrt[3]]*d*s*(s + r*x)*(Sqrt[(s^2 - r*s*x + r^2*x^2)/(
(1 + Sqrt[3])*s + r*x)^2]/(r^2*Sqrt[a + b*x^3]*Sqrt[s*((s + r*x)/((1 + Sqrt
[3])*s + r*x)^2)]))*EllipticE[ArcSin[((1 - Sqrt[3])*s + r*x)/((1 + Sqrt[3])
*s + r*x)], -7 - 4*Sqrt[3]], x]] /; FreeQ[{a, b, c, d}, x] && PosQ[a] && Eq
Q[b*c^3 - 2*(5 - 3*Sqrt[3])*a*d^3, 0]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4293
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x]]
*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
 Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

3.27. $\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx$

3.27.4 Maple [F]

$$\int \frac{\sqrt{a+a\csc\left(dx+c\right)}}{\csc\left(dx+c\right)^{\frac{4}{3}}}dx$$

input
```
int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(4/3),x)
```

output
```
int((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(4/3),x)
```

3.27.5 Fricas [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx = \int \frac{\sqrt{a\csc\left(dx+c\right)+a}}{\csc\left(dx+c\right)^{\frac{4}{3}}}\,dx$$

input
```
integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(4/3),x, algorithm="fricas")
```

output
```
integral(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(4/3), x)
```

3.27.6 Sympy [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx = \int \frac{\sqrt{a\left(\csc\left(c+dx\right)+1\right)}}{\csc^{\frac{4}{3}}\left(c+dx\right)}\,dx$$

input
```
integrate((a+a*csc(d*x+c))**(1/2)/csc(d*x+c)**(4/3),x)
```

output
```
Integral(sqrt(a*(csc(c + d*x) + 1))/csc(c + d*x)**(4/3), x)
```

3.27.7 Maxima [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx=\int \frac{\sqrt{a\csc\left(dx+c\right)+a}}{\csc\left(dx+c\right)^{\frac{4}{3}}}\,dx$$

input
```
integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(4/3),x, algorithm="maxima")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(4/3), x)
```

3.27.8 Giac [F]

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx=\int \frac{\sqrt{a\csc\left(dx+c\right)+a}}{\csc\left(dx+c\right)^{\frac{4}{3}}}\,dx$$

input
```
integrate((a+a*csc(d*x+c))^(1/2)/csc(d*x+c)^(4/3),x, algorithm="giac")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)/csc(d*x + c)^(4/3), x)
```

3.27.9 Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{a+a\csc(c+dx)}}{\csc^{\frac{4}{3}}(c+dx)}\,dx=\int \frac{\sqrt{a+\frac{a}{\sin(c+d\,x)}}}{{\left(\frac{1}{\sin(c+d\,x)}\right)}^{4/3}}\,dx$$

input
```
int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(4/3),x)
```

output
```
int((a + a/sin(c + d*x))^(1/2)/(1/sin(c + d*x))^(4/3), x)
```

3.28 $\int \csc^n(c+dx)\sqrt{a+a\csc(c+dx)}\,dx$

3.28.1 Optimal result

Integrand size = 23, antiderivative size = 48

$$\int \csc^n(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = -\frac{2a\cot(c+dx)\,\text{Hypergeometric2F1}\left(\frac{1}{2},1-n,\frac{3}{2},1-\csc(c+dx)\right)}{d\sqrt{a+a\csc(c+dx)}}$$

output
```
-2*a*cot(d*x+c)*hypergeom([1/2, 1-n],[3/2],1-csc(d*x+c))/d/(a+a*csc(d*x+c)
)^(1/2)
```

3.28.2 Mathematica [A] (verified)

Time = 1.18 (sec) , antiderivative size = 48, normalized size of antiderivative = 1.00

$$\int \csc^n(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = -\frac{2a\cot(c+dx)\,\text{Hypergeometric2F1}\left(\frac{1}{2},1-n,\frac{3}{2},1-\csc(c+dx)\right)}{d\sqrt{a(1+\csc(c+dx))}}$$

input
```
Integrate[Csc[c + d*x]^n*Sqrt[a + a*Csc[c + d*x]],x]
```

output
```
(-2*a*Cot[c + d*x]*Hypergeometric2F1[1/2, 1 - n, 3/2, 1 - Csc[c + d*x]])/(
d*Sqrt[a*(1 + Csc[c + d*x])])
```

3.28.3 Rubi [A] (verified)

Time = 0.26 (sec) , antiderivative size = 48, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.130, Rules used = {3042, 4293, 75}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sqrt{a\csc(c+dx)+a}\csc^n(c+dx)\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \sqrt{a\csc(c+dx)+a}\csc(c+dx)^n dx$$

$$\Big\downarrow \text{4293}$$

$$\frac{a^2\cot(c+dx)\int \frac{\csc^{n-1}(c+dx)}{\sqrt{a-a\csc(c+dx)}}d\csc(c+dx)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \text{75}$$

$$-\frac{2a\cot(c+dx)\text{Hypergeometric2F1}\left(\frac{1}{2},1-n,\frac{3}{2},1-\csc(c+dx)\right)}{d\sqrt{a\csc(c+dx)+a}}$$

input
```
Int[Csc[c + d*x]^n*Sqrt[a + a*Csc[c + d*x]],x]
```

output
```
(-2*a*Cot[c + d*x]*Hypergeometric2F1[1/2, 1 - n, 3/2, 1 - Csc[c + d*x]])/(
d*Sqrt[a + a*Csc[c + d*x]])
```

3.28.3.1 Defintions of rubi rules used

rule 75
```
Int[((b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[((c + d*x
)^(n + 1)/(d*(n + 1)*(-d/(b*c))^m))*Hypergeometric2F1[-m, n + 1, n + 2, 1 +
 d*(x/c)], x] /; FreeQ[{b, c, d, m, n}, x] &&  !IntegerQ[n] && (IntegerQ[m]
 || GtQ[-d/(b*c), 0])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4293
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x]]
*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
 Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

3.28.4 Maple [F]

$$\int \csc(dx+c)^n \sqrt{a+a\csc(dx+c)}dx$$

input
```
int(csc(d*x+c)^n*(a+a*csc(d*x+c))^(1/2),x)
```

output
```
int(csc(d*x+c)^n*(a+a*csc(d*x+c))^(1/2),x)
```

3.28.5 Fricas [F]

$$\int \csc^n(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a\csc(dx+c)+a}\csc(dx+c)^n\,dx$$

input
```
integrate(csc(d*x+c)^n*(a+a*csc(d*x+c))^(1/2),x, algorithm="fricas")
```

output
```
integral(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

3.28.6 Sympy [F]

$$\int \csc^n(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a(\csc(c+dx)+1)}\csc^n(c+dx)\,dx$$

input
```
integrate(csc(d*x+c)**n*(a+a*csc(d*x+c))**(1/2),x)
```

output
```
Integral(sqrt(a*(csc(c + d*x) + 1))*csc(c + d*x)**n, x)
```

3.28.7 Maxima [F]

$$\int \csc^n(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a\csc(dx+c)+a}\csc(dx+c)^n\,dx$$

input
```
integrate(csc(d*x+c)^n*(a+a*csc(d*x+c))^(1/2),x, algorithm="maxima")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

3.28.8 Giac [F]

$$\int \csc^n(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a\csc(dx+c)+a}\csc(dx+c)^n\,dx$$

input
```
integrate(csc(d*x+c)^n*(a+a*csc(d*x+c))^(1/2),x, algorithm="giac")
```

output
```
integrate(sqrt(a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

3.28.9 Mupad [F(-1)]

Timed out.

$$\int \csc^n(c+dx)\sqrt{a+a\csc(c+dx)}\,dx = \int \sqrt{a+\frac{a}{\sin(c+d\,x)}}\left(\frac{1}{\sin(c+d\,x)}\right)^n dx$$

input
```
int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^n,x)
```

output
```
int((a + a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^n, x)
```

3.29 $\int \csc^n(c+dx)\sqrt{a-a\csc(c+dx)}\,dx$

3.29.1 Optimal result

Integrand size = 24, antiderivative size = 69

$$\int \csc^n(c+dx)\sqrt{a-a\csc(c+dx)}\,dx = -\frac{2a\cos(c+dx)(-\csc(c+dx))^{-n}\csc^{1+n}(c+dx)\,\text{Hypergeometric2F1}\left(\frac{1}{2},1-n,\frac{3}{2},1+\csc(c+dx)\right)}{d\sqrt{a-a\csc(c+dx)}}$$

output
```
-2*a*cos(d*x+c)*csc(d*x+c)^(1+n)*hypergeom([1/2, 1-n],[3/2],1+csc(d*x+c))/
d/((-csc(d*x+c))^n)/(a-a*csc(d*x+c))^(1/2)
```

3.29.2 Mathematica [A] (verified)

Time = 2.90 (sec) , antiderivative size = 73, normalized size of antiderivative = 1.06

$$\int \csc^n(c+dx)\sqrt{a-a\csc(c+dx)}\,dx = -\frac{2a\cos(c+dx)\csc^{1+2n}(c+dx)\left(-\csc^2(c+dx)\right)^{-n}\text{Hypergeometric2F1}\left(\frac{1}{2},1-n,\frac{3}{2},1+\csc(c+dx)\right)}{d\sqrt{a-a\csc(c+dx)}}$$

input
```
Integrate[Csc[c + d*x]^n*Sqrt[a - a*Csc[c + d*x]],x]
```

output
```
(-2*a*Cos[c + d*x]*Csc[c + d*x]^(1 + 2*n)*Hypergeometric2F1[1/2, 1 - n, 3/
2, 1 + Csc[c + d*x]])/(d*(-Csc[c + d*x]^2)^n*Sqrt[a - a*Csc[c + d*x]])
```

3.29.3 Rubi [A] (verified)

Time = 0.26 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.167, Rules used = {3042, 4293, 77, 75}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sqrt{a-a\csc(c+dx)}\csc^n(c+dx)\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \sqrt{a-a\csc(c+dx)}\csc(c+dx)^n dx$$

$$\Big\downarrow \text{4293}$$

$$\frac{a^2\cot(c+dx)\int \frac{\csc^{n-1}(c+dx)}{\sqrt{\csc(c+dx)a+a}}d\csc(c+dx)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \text{77}$$

$$-\frac{a^2\cos(c+dx)(-\csc(c+dx))^{-n}\csc^{n+1}(c+dx)\int \frac{(-\csc(c+dx))^{n-1}}{\sqrt{\csc(c+dx)a+a}}d\csc(c+dx)}{d\sqrt{a-a\csc(c+dx)}\sqrt{a\csc(c+dx)+a}}$$

$$\Big\downarrow \text{75}$$

$$-\frac{2a\cos(c+dx)(-\csc(c+dx))^{-n}\csc^{n+1}(c+dx)\,\text{Hypergeometric2F1}\left(\frac{1}{2},1-n,\frac{3}{2},\csc(c+dx)+1\right)}{d\sqrt{a-a\csc(c+dx)}}$$

input
```
Int[Csc[c + d*x]^n*Sqrt[a - a*Csc[c + d*x]],x]
```

output
```
(-2*a*Cos[c + d*x]*Csc[c + d*x]^(1 + n)*Hypergeometric2F1[1/2, 1 - n, 3/2,
 1 + Csc[c + d*x]])/(d*(-Csc[c + d*x])^n*Sqrt[a - a*Csc[c + d*x]])
```

3.29.3.1 Defintions of rubi rules used

rule 75
```
Int[((b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[((c + d*x
)^(n + 1)/(d*(n + 1)*(-d/(b*c))^m))*Hypergeometric2F1[-m, n + 1, n + 2, 1 +
 d*(x/c)], x] /; FreeQ[{b, c, d, m, n}, x] &&  !IntegerQ[n] && (IntegerQ[m]
 || GtQ[-d/(b*c), 0])
```

rule 77
```
Int[((b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[((-b)*(c/
d))^IntPart[m]*((b*x)^FracPart[m]/((-d)*(x/c))^FracPart[m])   Int[((-d)*(x/
c))^m*(c + d*x)^n, x], x] /; FreeQ[{b, c, d, m, n}, x] &&  !IntegerQ[m] &&
 !IntegerQ[n] &&  !GtQ[c, 0] &&  !GtQ[-d/(b*c), 0]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4293
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*Sqrt[csc[(e_.) + (f_.)*(x_)]*(b_.)
 + (a_)], x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x]]
*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)/Sqrt[a - b*x], x], x,
 Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, n}, x] && EqQ[a^2 - b^2, 0]
```

3.29.4 Maple [F]

$$\int \csc{(dx + c)}^{n} \sqrt{a - a\csc{(dx + c)}}dx$$

input
```
int(csc(d*x+c)^n*(a-a*csc(d*x+c))^(1/2),x)
```

output
```
int(csc(d*x+c)^n*(a-a*csc(d*x+c))^(1/2),x)
```

3.29.5 Fricas [F]

$$\int \csc^n(c+dx)\sqrt{a-a\csc(c+dx)}\,dx=\int \sqrt{-a\csc\left(dx+c\right)+a}\csc\left(dx+c\right)^n\,dx$$

input
```
integrate(csc(d*x+c)^n*(a-a*csc(d*x+c))^(1/2),x, algorithm="fricas")
```

output
```
integral(sqrt(-a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

3.29.6 Sympy [F]

$$\int \csc^n(c+dx)\sqrt{a-a\csc(c+dx)}\,dx=\int \sqrt{-a\left(\csc\left(c+dx\right)-1\right)}\csc^n\left(c+dx\right)dx$$

input
```
integrate(csc(d*x+c)**n*(a-a*csc(d*x+c))**(1/2),x)
```

output
```
Integral(sqrt(-a*(csc(c + d*x) - 1))*csc(c + d*x)**n, x)
```

3.29.7 Maxima [F]

$$\int \csc^n(c+dx)\sqrt{a-a\csc(c+dx)}\,dx=\int \sqrt{-a\csc\left(dx+c\right)+a}\csc\left(dx+c\right)^n\,dx$$

input
```
integrate(csc(d*x+c)^n*(a-a*csc(d*x+c))^(1/2),x, algorithm="maxima")
```

output
```
integrate(sqrt(-a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

3.29.8 Giac [F]

$$\int \csc^n(c+dx)\sqrt{a-a\csc(c+dx)}\,dx = \int \sqrt{-a\csc\left(dx+c\right)+a}\csc\left(dx+c\right)^n\,dx$$

input
```
integrate(csc(d*x+c)^n*(a-a*csc(d*x+c))^(1/2),x, algorithm="giac")
```

output
```
integrate(sqrt(-a*csc(d*x + c) + a)*csc(d*x + c)^n, x)
```

3.29.9 Mupad [F(-1)]

Timed out.

$$\int \csc^n(c+dx)\sqrt{a-a\csc(c+dx)}\,dx = \int \sqrt{a-\frac{a}{\sin\left(c+d\,x\right)}}\left(\frac{1}{\sin\left(c+d\,x\right)}\right)^n\,dx$$

input
```
int((a - a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^n,x)
```

output
```
int((a - a/sin(c + d*x))^(1/2)*(1/sin(c + d*x))^n, x)
```

3.30 $\int \csc^3(e+fx)(a+a\csc(e+fx))^m\,dx$

3.30.1 Optimal result

Integrand size = 21, antiderivative size = 156

$$\int \csc^3(e+fx)(a+a\csc(e+fx))^m\,dx$$
$$=\frac{\cot(e+fx)(a+a\csc(e+fx))^m}{f(2+3m+m^2)}-\frac{\cot(e+fx)(a+a\csc(e+fx))^{1+m}}{af(2+m)}$$
$$-\frac{2^{\frac{1}{2}+m}(1+m+m^2)\cot(e+fx)(1+\csc(e+fx))^{-\frac{1}{2}-m}(a+a\csc(e+fx))^m\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{1}{2}-\right.}{f(1+m)(2+m)}$$

output

```
cot(f*x+e)*(a+a*csc(f*x+e))^m/f/(m^2+3*m+2)-cot(f*x+e)*(a+a*csc(f*x+e))^(1
+m)/a/f/(2+m)-2^(1/2+m)*(m^2+m+1)*cot(f*x+e)*(1+csc(f*x+e))^(-1/2-m)*(a+a*
csc(f*x+e))^m*hypergeom([1/2, 1/2-m],[3/2],1/2-1/2*csc(f*x+e))/f/(m^2+3*m+
2)
```

3.30.2 Mathematica [A] (verified)

Time = 5.54 (sec) , antiderivative size = 178, normalized size of antiderivative = 1.14

$$\int \csc^3(e+fx)(a+a\csc(e+fx))^m\,dx =$$
$$-\frac{(a(1+\csc(e+fx)))^m\left((-2+m)m\cot^4\left(\frac{1}{2}(e+fx)\right)\text{Hypergeometric2F1}\left(-2-m,-2m,-1-m,-\text{t}\right.\right.}{}$$

input
```
Integrate[Csc[e + f*x]^3*(a + a*Csc[e + f*x])^m,x]
```

output
```
-1/4*((a*(1 + Csc[e + f*x]))^m*((-2 + m)*m*Cot[(e + f*x)/2]^4*Hypergeometr
ic2F1[-2 - m, -2*m, -1 - m, -Tan[(e + f*x)/2]] + (2 + m)*(m*Hypergeometric
2F1[2 - m, -2*m, 3 - m, -Tan[(e + f*x)/2]] + 2*(-2 + m)*Cot[(e + f*x)/2]^2
*Hypergeometric2F1[-2*m, -m, 1 - m, -Tan[(e + f*x)/2]]))*Tan[(e + f*x)/2]^
2)/(f*(-2 + m)*m*(2 + m)*(1 + Tan[(e + f*x)/2])^(2*m))
```

3.30.3 Rubi [A] (verified)

Time = 0.67 (sec) , antiderivative size = 158, normalized size of antiderivative = 1.01, number of steps used = 10, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}} = 0.429$, Rules used = {3042, 4287, 3042, 4489, 3042, 4315, 3042, 4314, 79}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^3(e+fx)(a\csc(e+fx)+a)^m\,dx$$

$$\Big\downarrow \ 3042$$

$$\int \csc(e+fx)^3(a\csc(e+fx)+a)^m dx$$

$$\Big\downarrow \ 4287$$

$$\frac{\int \csc(e+fx)(a(m+1)-a\csc(e+fx))(\csc(e+fx)a+a)^m dx}{a(m+2)}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^{m+1}}{af(m+2)}$$

$$\Big\downarrow \ 3042$$

$$\frac{\int \csc(e+fx)(a(m+1)-a\csc(e+fx))(\csc(e+fx)a+a)^m dx}{a(m+2)}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^{m+1}}{af(m+2)}$$

$$\Big\downarrow \ 4489$$

$$\frac{\frac{a(m^2+m+1)\int \csc(e+fx)(\csc(e+fx)a+a)^m dx}{m+1}+\frac{a\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}}{a(m+2)}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^{m+1}}{af(m+2)}$$

$\Big\downarrow$ 3042

$$\frac{\frac{a(m^2+m+1)\int \csc(e+fx)(\csc(e+fx)a+a)^m dx}{m+1}+\frac{a\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}}{a(m+2)}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^{m+1}}{af(m+2)}$$

$\Big\downarrow$ 4315

$$\frac{\frac{a(m^2+m+1)(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int \csc(e+fx)(\csc(e+fx)+1)^m dx}{m+1}+\frac{a\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}}{a(m+2)}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^{m+1}}{af(m+2)}$$

$\Big\downarrow$ 3042

$$\frac{\frac{a(m^2+m+1)(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int \csc(e+fx)(\csc(e+fx)+1)^m dx}{m+1}+\frac{a\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}}{a(m+2)}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^{m+1}}{af(m+2)}$$

$\Big\downarrow$ 4314

$$\frac{\frac{a(m^2+m+1)\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m\int \frac{(\csc(e+fx)+1)^{m-\frac{1}{2}}}{\sqrt{1-\csc(e+fx)}}d\csc(e+fx)}{f(m+1)\sqrt{1-\csc(e+fx)}}+\frac{a\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}}{a(m+2)}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^{m+1}}{af(m+2)}$$

$\Big\downarrow$ 79

$$\frac{\frac{a\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}-\frac{a2^{m+\frac{1}{2}}(m^2+m+1)\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m \text{Hypergeometric2F1}\left(\frac{1}{2},\frac{1}{2}-m,\frac{3}{2},\frac{1}{2}(1-\right.}{f(m+1)}}{a(m+2)}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^{m+1}}{af(m+2)}$$

input
```
Int[Csc[e + f*x]^3*(a + a*Csc[e + f*x])^m,x]
```

3.30. $\int \csc^3(e+fx)(a+a\csc(e+fx))^m\,dx$

output
```
-((Cot[e + f*x]*(a + a*Csc[e + f*x])^(1 + m))/(a*f*(2 + m))) + ((a*Cot[e +
 f*x]*(a + a*Csc[e + f*x])^m)/(f*(1 + m)) - (2^(1/2 + m)*a*(1 + m + m^2)*C
ot[e + f*x]*(1 + Csc[e + f*x])^(-1/2 - m)*(a + a*Csc[e + f*x])^m*Hypergeom
etric2F1[1/2, 1/2 - m, 3/2, (1 - Csc[e + f*x])/2])/(f*(1 + m)))/(a*(2 + m)
)
```

3.30.3.1 Defintions of rubi rules used

rule 79
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[((
a + b*x)^(m + 1)/(b*(m + 1)*(b/(b*c - a*d))^n))*Hypergeometric2F1[-n, m + 1
, m + 2, (-d)*((a + b*x)/(b*c - a*d))], x] /; FreeQ[{a, b, c, d, m, n}, x]
&&  !IntegerQ[m] &&  !IntegerQ[n] && GtQ[b/(b*c - a*d), 0] && (RationalQ[m]
 ||  !(RationalQ[n] && GtQ[-d/(b*c - a*d), 0]))
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4287
```
Int[csc[(e_.) + (f_.)*(x_)]^3*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_),
x_Symbol] :> Simp[(-Cot[e + f*x])*((a + b*Csc[e + f*x])^(m + 1)/(b*f*(m + 2
))), x] + Simp[1/(b*(m + 2))   Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m*(b*(
m + 1) - a*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f, m}, x] && EqQ[a^2 -
b^2, 0] &&  !LtQ[m, -2^(-1)]
```

rule 4314
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x
]]*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)*((a + b*x)^(m - 1/2
)/Sqrt[a - b*x]), x], x, Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, m, n},
x] && EqQ[a^2 - b^2, 0] &&  !IntegerQ[m] && GtQ[a, 0]
```

rule 4315
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Simp[a^IntPart[m]*((a + b*Csc[e + f*x])^FracPart[m
]/(1 + (b/a)*Csc[e + f*x])^FracPart[m])   Int[(1 + (b/a)*Csc[e + f*x])^m*(d
*Csc[e + f*x])^n, x], x] /; FreeQ[{a, b, d, e, f, m, n}, x] && EqQ[a^2 - b^
2, 0] &&  !IntegerQ[m] &&  !GtQ[a, 0]
```

rule 4489
```
Int[csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_)*(cs
c[(e_.) + (f_.)*(x_)]*(B_.) + (A_)), x_Symbol] :> Simp[(-B)*Cot[e + f*x]*((
a + b*Csc[e + f*x])^m/(f*(m + 1))), x] + Simp[(a*B*m + A*b*(m + 1))/(b*(m +
 1))   Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m, x], x] /; FreeQ[{a, b, A, B
, e, f, m}, x] && NeQ[A*b - a*B, 0] && EqQ[a^2 - b^2, 0] && NeQ[a*B*m + A*b
*(m + 1), 0] &&  !LtQ[m, -2^(-1)]
```

3.30.4 Maple [F]

$$\int \csc(fx+e)^3 (a+a\csc(fx+e))^m\,dx$$

input
```
int(csc(f*x+e)^3*(a+a*csc(f*x+e))^m,x)
```

output
```
int(csc(f*x+e)^3*(a+a*csc(f*x+e))^m,x)
```

3.30.5 Fricas [F]

$$\int \csc^3(e+fx)(a+a\csc(e+fx))^m\,dx = \int (a\csc(fx+e)+a)^m \csc(fx+e)^3\,dx$$

input
```
integrate(csc(f*x+e)^3*(a+a*csc(f*x+e))^m,x, algorithm="fricas")
```

output
```
integral((a*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

3.30.6 Sympy [F]

$$\int \csc^3(e+fx)(a+a\csc(e+fx))^m\,dx = \int (a(\csc(e+fx)+1))^m \csc^3(e+fx)\,dx$$

input
```
integrate(csc(f*x+e)**3*(a+a*csc(f*x+e))**m,x)
```

output
```
Integral((a*(csc(e + f*x) + 1))**m*csc(e + f*x)**3, x)
```

3.30.7 Maxima [F]

$$\int \csc^3(e+fx)(a+a\csc(e+fx))^m\,dx = \int (a\csc(fx+e)+a)^m \csc(fx+e)^3\,dx$$

input
```
integrate(csc(f*x+e)^3*(a+a*csc(f*x+e))^m,x, algorithm="maxima")
```

output
```
integrate((a*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

3.30.8 Giac [F]

$$\int \csc^3(e+fx)(a+a\csc(e+fx))^m\,dx = \int (a\csc(fx+e)+a)^m \csc(fx+e)^3\,dx$$

input
```
integrate(csc(f*x+e)^3*(a+a*csc(f*x+e))^m,x, algorithm="giac")
```

output
```
integrate((a*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

3.30.9 Mupad [F(-1)]

Timed out.

$$\int \csc^3(e+fx)(a+a\csc(e+fx))^m\,dx = \int \frac{\left(a+\frac{a}{\sin(e+f\,x)}\right)^m}{\sin(e+f\,x)^3}\,dx$$

input
```
int((a + a/sin(e + f*x))^m/sin(e + f*x)^3,x)
```

output
```
int((a + a/sin(e + f*x))^m/sin(e + f*x)^3, x)
```

3.31 $\int \csc^2(e+fx)(a+a\csc(e+fx))^m\,dx$

3.31.1 Optimal result

Integrand size = 21, antiderivative size = 109

$$\int \csc^2(e+fx)(a+a\csc(e+fx))^m\,dx = -\frac{\cot(e+fx)(a+a\csc(e+fx))^m}{f(1+m)} - \frac{2^{\frac{1}{2}+m}m\cot(e+fx)(1+\csc(e+fx))^{-\frac{1}{2}-m}(a+a\csc(e+fx))^m \text{Hypergeometric2F1}\left(\frac{1}{2},\frac{1}{2}-m,\frac{3}{2},\frac{1}{2}(1-\right.}{f(1+m)}$$

output
```
-cot(f*x+e)*(a+a*csc(f*x+e))^m/f/(1+m)-2^(1/2+m)*m*cot(f*x+e)*(1+csc(f*x+e
))^(-1/2-m)*(a+a*csc(f*x+e))^m*hypergeom([1/2, 1/2-m],[3/2],1/2-1/2*csc(f*
x+e))/f/(1+m)
```

3.31.2 Mathematica [A] (verified)

Time = 0.97 (sec) , antiderivative size = 126, normalized size of antiderivative = 1.16

$$\int \csc^2(e+fx)(a+a\csc(e+fx))^m\,dx = -\frac{(a(1+\csc(e+fx)))^m\left((-1+m)\cot^2\left(\frac{1}{2}(e+fx)\right)\text{Hypergeometric2F1}\left(-1-m,-2m,-m,-\tan\left(\frac{1}{2}(\right.\right.\right.}{}$$

input
```
Integrate[Csc[e + f*x]^2*(a + a*Csc[e + f*x])^m,x]
```

output
```
-1/2*((a*(1 + Csc[e + f*x]))^m*((-1 + m)*Cot[(e + f*x)/2]^2*Hypergeometric
2F1[-1 - m, -2*m, -m, -Tan[(e + f*x)/2]] + (1 + m)*Hypergeometric2F1[1 - m
, -2*m, 2 - m, -Tan[(e + f*x)/2]])*Tan[(e + f*x)/2])/(f*(-1 + m)*(1 + m)*(
1 + Tan[(e + f*x)/2])^(2*m))
```

3.31.3 Rubi [A] (verified)

Time = 0.43 (sec) , antiderivative size = 109, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$, Rules used = {3042, 4285, 3042, 4315, 3042, 4314, 79}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^2(e+fx)(a\csc(e+fx)+a)^m\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \csc(e+fx)^2(a\csc(e+fx)+a)^m dx$$

$$\Big\downarrow \text{4285}$$

$$\frac{m\int \csc(e+fx)(\csc(e+fx)a+a)^m dx}{m+1}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}$$

$$\Big\downarrow \text{3042}$$

$$\frac{m\int \csc(e+fx)(\csc(e+fx)a+a)^m dx}{m+1}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}$$

$$\Big\downarrow \text{4315}$$

$$\frac{m(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int \csc(e+fx)(\csc(e+fx)+1)^m dx}{m+1}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}$$

$$\Big\downarrow \text{3042}$$

$$\frac{m(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int \csc(e+fx)(\csc(e+fx)+1)^m dx}{m+1}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}$$

$$\Big\downarrow \text{4314}$$

$$\frac{m\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m\int\frac{(\csc(e+fx)+1)^{m-\frac{1}{2}}}{\sqrt{1-\csc(e+fx)}}d\csc(e+fx)}{f(m+1)\sqrt{1-\csc(e+fx)}}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}$$

$\big\downarrow$ 79

$$-\frac{2^{m+\frac{1}{2}}m\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{1}{2}-m,\frac{3}{2},\frac{1}{2}(1-\csc(e+\right.}{f(m+1)}-\frac{\cot(e+fx)(a\csc(e+fx)+a)^m}{f(m+1)}$$

input
```
Int[Csc[e + f*x]^2*(a + a*Csc[e + f*x])^m,x]
```

output
```
-((Cot[e + f*x]*(a + a*Csc[e + f*x])^m)/(f*(1 + m))) - (2^(1/2 + m)*m*Cot[
e + f*x]*(1 + Csc[e + f*x])^(-1/2 - m)*(a + a*Csc[e + f*x])^m*Hypergeometr
ic2F1[1/2, 1/2 - m, 3/2, (1 - Csc[e + f*x])/2])/(f*(1 + m))
```

3.31.3.1 Defintions of rubi rules used

rule 79
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[((
a + b*x)^(m + 1)/(b*(m + 1)*(b/(b*c - a*d))^n))*Hypergeometric2F1[-n, m + 1
, m + 2, (-d)*((a + b*x)/(b*c - a*d))], x] /; FreeQ[{a, b, c, d, m, n}, x]
&&  !IntegerQ[m] &&  !IntegerQ[n] && GtQ[b/(b*c - a*d), 0] && (RationalQ[m]
 ||  !(RationalQ[n] && GtQ[-d/(b*c - a*d), 0]))
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4285
```
Int[csc[(e_.) + (f_.)*(x_)]^2*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_),
x_Symbol] :> Simp[(-Cot[e + f*x])*((a + b*Csc[e + f*x])^m/(f*(m + 1))), x]
+ Simp[a*(m/(b*(m + 1)))   Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m, x], x]
/; FreeQ[{a, b, e, f, m}, x] && EqQ[a^2 - b^2, 0] &&  !LtQ[m, -2^(-1)]
```

rule 4314
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x
]]*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)*((a + b*x)^(m - 1/2
)/Sqrt[a - b*x]), x], x, Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, m, n},
x] && EqQ[a^2 - b^2, 0] &&  !IntegerQ[m] && GtQ[a, 0]
```

rule 4315
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Simp[a^IntPart[m]*((a + b*Csc[e + f*x])^FracPart[m
]/(1 + (b/a)*Csc[e + f*x])^FracPart[m])   Int[(1 + (b/a)*Csc[e + f*x])^m*(d
*Csc[e + f*x])^n, x], x] /; FreeQ[{a, b, d, e, f, m, n}, x] && EqQ[a^2 - b^
2, 0] &&  !IntegerQ[m] &&  !GtQ[a, 0]
```

3.31.4 Maple [F]

$$\int \csc \left(fx+e\right)^{2} \left(a+a\csc \left(fx+e\right)\right)^{m}dx$$

input
```
int(csc(f*x+e)^2*(a+a*csc(f*x+e))^m,x)
```

output
```
int(csc(f*x+e)^2*(a+a*csc(f*x+e))^m,x)
```

3.31.5 Fricas [F]

$$\int \csc^2(e+fx)(a+a\csc(e+fx))^m\,dx = \int \left(a\csc \left(fx+e\right)+a\right)^{m}\csc \left(fx+e\right)^{2}\,dx$$

input
```
integrate(csc(f*x+e)^2*(a+a*csc(f*x+e))^m,x, algorithm="fricas")
```

output
```
integral((a*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

3.31.6 Sympy [F]

$$\int \csc^2(e+fx)(a+a\csc(e+fx))^m\,dx = \int (a(\csc(e+fx)+1))^m \csc^2(e+fx)\,dx$$

input
```
integrate(csc(f*x+e)**2*(a+a*csc(f*x+e))**m,x)
```

output
```
Integral((a*(csc(e + f*x) + 1))**m*csc(e + f*x)**2, x)
```

3.31.7 Maxima [F]

$$\int \csc^2(e+fx)(a+a\csc(e+fx))^m\,dx = \int (a\csc(fx+e)+a)^m \csc(fx+e)^2\,dx$$

input
```
integrate(csc(f*x+e)^2*(a+a*csc(f*x+e))^m,x, algorithm="maxima")
```

output
```
integrate((a*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

3.31.8 Giac [F]

$$\int \csc^2(e+fx)(a+a\csc(e+fx))^m\,dx = \int (a\csc(fx+e)+a)^m \csc(fx+e)^2\,dx$$

input
```
integrate(csc(f*x+e)^2*(a+a*csc(f*x+e))^m,x, algorithm="giac")
```

output
```
integrate((a*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

3.31.9 Mupad [F(-1)]

Timed out.

$$\int \csc^2(e+fx)(a+a\csc(e+fx))^m\,dx = \int \frac{\left(a+\frac{a}{\sin(e+f\,x)}\right)^m}{\sin\left(e+f\,x\right)^2}\,dx$$

input
```
int((a + a/sin(e + f*x))^m/sin(e + f*x)^2,x)
```

output
```
int((a + a/sin(e + f*x))^m/sin(e + f*x)^2, x)
```

3.32 $\int \csc(e+fx)(a+a\csc(e+fx))^m\,dx$

3.32.1 Optimal result

Integrand size = 19, antiderivative size = 74

$$\int \csc(e+fx)(a+a\csc(e+fx))^m\,dx =$$
$$-\frac{2^{\frac{1}{2}+m}\cot(e+fx)(1+\csc(e+fx))^{-\frac{1}{2}-m}(a+a\csc(e+fx))^m\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{1}{2}-m,\frac{3}{2},\frac{1}{2}(1-c\right.}{f}$$

output
```
-2^(1/2+m)*cot(f*x+e)*(1+csc(f*x+e))^(-1/2-m)*(a+a*csc(f*x+e))^m*hypergeom
([1/2, 1/2-m],[3/2],1/2-1/2*csc(f*x+e))/f
```

3.32.2 Mathematica [A] (verified)

Time = 0.38 (sec) , antiderivative size = 60, normalized size of antiderivative = 0.81

$$\int \csc(e+fx)(a+a\csc(e+fx))^m\,dx =$$
$$-\frac{(a(1+\csc(e+fx)))^m\,\text{Hypergeometric2F1}\left(-2m,-m,1-m,-\tan\left(\frac{1}{2}(e+fx)\right)\right)\left(1+\tan\left(\frac{1}{2}(e+fx\right.\right.}{fm}$$

input
```
Integrate[Csc[e + f*x]*(a + a*Csc[e + f*x])^m,x]
```

output
```
-(((a*(1 + Csc[e + f*x]))^m*Hypergeometric2F1[-2*m, -m, 1 - m, -Tan[(e + f
*x)/2]])/(f*m*(1 + Tan[(e + f*x)/2])^(2*m)))
```

3.32.3 Rubi [A] (verified)

Time = 0.30 (sec) , antiderivative size = 74, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.263$, Rules used = {3042, 4315, 3042, 4314, 79}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc(e+fx)(a\csc(e+fx)+a)^m\,dx$$

$$\Big\downarrow \text{ 3042}$$

$$\int \csc(e+fx)(a\csc(e+fx)+a)^m dx$$

$$\Big\downarrow \text{ 4315}$$

$$(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int \csc(e+fx)(\csc(e+fx)+1)^m dx$$

$$\Big\downarrow \text{ 3042}$$

$$(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int \csc(e+fx)(\csc(e+fx)+1)^m dx$$

$$\Big\downarrow \text{ 4314}$$

$$\frac{\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m\int \frac{(\csc(e+fx)+1)^{m-\frac{1}{2}}}{\sqrt{1-\csc(e+fx)}}d\csc(e+fx)}{f\sqrt{1-\csc(e+fx)}}$$

$$\Big\downarrow \text{ 79}$$

$$-\frac{2^{m+\frac{1}{2}}\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m \text{Hypergeometric2F1}\left(\frac{1}{2},\frac{1}{2}-m,\frac{3}{2},\frac{1}{2}(1-\csc(e+f}{f}$$

input
```
Int[Csc[e + f*x]*(a + a*Csc[e + f*x])^m,x]
```

output
```
-((2^(1/2 + m)*Cot[e + f*x]*(1 + Csc[e + f*x])^(-1/2 - m)*(a + a*Csc[e + f
*x])^m*Hypergeometric2F1[1/2, 1/2 - m, 3/2, (1 - Csc[e + f*x])/2])/f)
```

3.32.3.1 Defintions of rubi rules used

rule 79
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[((
a + b*x)^(m + 1)/(b*(m + 1)*(b/(b*c - a*d))^n))*Hypergeometric2F1[-n, m + 1
, m + 2, (-d)*((a + b*x)/(b*c - a*d))], x] /; FreeQ[{a, b, c, d, m, n}, x]
&&  !IntegerQ[m] &&  !IntegerQ[n] && GtQ[b/(b*c - a*d), 0] && (RationalQ[m]
 ||  !(RationalQ[n] && GtQ[-d/(b*c - a*d), 0]))
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4314
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x
]]*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)*((a + b*x)^(m - 1/2
)/Sqrt[a - b*x]), x], x, Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, m, n},
x] && EqQ[a^2 - b^2, 0] &&  !IntegerQ[m] && GtQ[a, 0]
```

rule 4315
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Simp[a^IntPart[m]*((a + b*Csc[e + f*x])^FracPart[m
]/(1 + (b/a)*Csc[e + f*x])^FracPart[m])   Int[(1 + (b/a)*Csc[e + f*x])^m*(d
*Csc[e + f*x])^n, x], x] /; FreeQ[{a, b, d, e, f, m, n}, x] && EqQ[a^2 - b^
2, 0] &&  !IntegerQ[m] &&  !GtQ[a, 0]
```

3.32.4 Maple [F]

$$\int \csc(fx+e)\,(a+a\csc(fx+e))^m\,dx$$

input
```
int(csc(f*x+e)*(a+a*csc(f*x+e))^m,x)
```

output
```
int(csc(f*x+e)*(a+a*csc(f*x+e))^m,x)
```

3.32.5 Fricas [F]

$$\int \csc(e+fx)(a+a\csc(e+fx))^m\,dx = \int (a\csc(fx+e)+a)^m\csc(fx+e)\,dx$$

input
```
integrate(csc(f*x+e)*(a+a*csc(f*x+e))^m,x, algorithm="fricas")
```

output
```
integral((a*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

3.32.6 Sympy [F]

$$\int \csc(e+fx)(a+a\csc(e+fx))^m\,dx = \int (a(\csc(e+fx)+1))^m\csc(e+fx)\,dx$$

input
```
integrate(csc(f*x+e)*(a+a*csc(f*x+e))**m,x)
```

output
```
Integral((a*(csc(e + f*x) + 1))**m*csc(e + f*x), x)
```

3.32.7 Maxima [F]

$$\int \csc(e+fx)(a+a\csc(e+fx))^m\,dx = \int (a\csc(fx+e)+a)^m\csc(fx+e)\,dx$$

input
```
integrate(csc(f*x+e)*(a+a*csc(f*x+e))^m,x, algorithm="maxima")
```

output
```
integrate((a*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

3.32.8 Giac [F]

$$\int \csc(e+fx)(a+a\csc(e+fx))^m\,dx = \int (a\csc(fx+e)+a)^m \csc(fx+e)\ dx$$

input
```
integrate(csc(f*x+e)*(a+a*csc(f*x+e))^m,x, algorithm="giac")
```

output
```
integrate((a*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

3.32.9 Mupad [F(-1)]

Timed out.

$$\int \csc(e+fx)(a+a\csc(e+fx))^m\,dx = \int \frac{\left(a+\frac{a}{\sin(e+f\,x)}\right)^m}{\sin\left(e+f\,x\right)}\,dx$$

input
```
int((a + a/sin(e + f*x))^m/sin(e + f*x),x)
```

output
```
int((a + a/sin(e + f*x))^m/sin(e + f*x), x)
```

3.33 $\int (a+a\csc(e+fx))^m\,dx$

3.33.1 Optimal result

Integrand size = 12, antiderivative size = 84

$$\int (a+a\csc(e+fx))^m\,dx = -\frac{\sqrt{2}\,\mathrm{AppellF1}\left(\frac{1}{2}+m,\frac{1}{2},1,\frac{3}{2}+m,\frac{1}{2}(1+\csc(e+fx)),1+\csc(e+fx)\right)\cot(e+fx)(a+a\csc(e+fx))^m}{f(1+2m)\sqrt{1-\csc(e+fx)}}$$

output
```
-AppellF1(1/2+m,1,1/2,3/2+m,1+csc(f*x+e),1/2+1/2*csc(f*x+e))*cot(f*x+e)*(a
+a*csc(f*x+e))^m*2^(1/2)/f/(1+2*m)/(1-csc(f*x+e))^(1/2)
```

3.33.2 Mathematica [F]

$$\int (a+a\csc(e+fx))^m\,dx = \int (a+a\csc(e+fx))^m\,dx$$

input
```
Integrate[(a + a*Csc[e + f*x])^m,x]
```

output
```
Integrate[(a + a*Csc[e + f*x])^m, x]
```

3.33.3 Rubi [A] (verified)

Time = 0.29 (sec) , antiderivative size = 84, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.417, Rules used = {3042, 4266, 3042, 4265, 153}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (a\csc(e+fx)+a)^m\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int (a\csc(e+fx)+a)^m dx$$

$$\Big\downarrow \text{4266}$$

$$(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int (\csc(e+fx)+1)^m dx$$

$$\Big\downarrow \text{3042}$$

$$(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int (\csc(e+fx)+1)^m dx$$

$$\Big\downarrow \text{4265}$$

$$\frac{\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m\int \frac{(\csc(e+fx)+1)^{m-\frac{1}{2}}\sin(e+fx)}{\sqrt{1-\csc(e+fx)}}d\csc(e+fx)}{f\sqrt{1-\csc(e+fx)}}$$

$$\Big\downarrow \text{153}$$

$$-\frac{\sqrt{2}\cot(e+fx)(a\csc(e+fx)+a)^m \operatorname{AppellF1}\left(m+\frac{1}{2},\frac{1}{2},1,m+\frac{3}{2},\frac{1}{2}(\csc(e+fx)+1),\csc(e+fx)+1\right)}{f(2m+1)\sqrt{1-\csc(e+fx)}}$$

input
```
Int[(a + a*Csc[e + f*x])^m,x]
```

output
```
-((Sqrt[2]*AppellF1[1/2 + m, 1/2, 1, 3/2 + m, (1 + Csc[e + f*x])/2, 1 + Cs
c[e + f*x]]*Cot[e + f*x]*(a + a*Csc[e + f*x])^m)/(f*(1 + 2*m)*Sqrt[1 - Csc
[e + f*x]]))
```

3.33.3.1 Defintions of rubi rules used

rule 153
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[(b*e - a*f)^p*((a + b*x)^(m + 1)/(b^(p + 1)*(m + 1)*Simp
lify[b/(b*c - a*d)]^n))*AppellF1[m + 1, -n, -p, m + 2, (-d)*((a + b*x)/(b*c
 - a*d)), (-f)*((a + b*x)/(b*e - a*f))], x] /; FreeQ[{a, b, c, d, e, f, m,
n}, x] &&  !IntegerQ[m] &&  !IntegerQ[n] && IntegerQ[p] && GtQ[Simplify[b/(
b*c - a*d)], 0] &&  !(GtQ[Simplify[d/(d*a - c*b)], 0] && SimplerQ[c + d*x,
a + b*x])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4265
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[a^n*(Cot
[c + d*x]/(d*Sqrt[1 + Csc[c + d*x]]*Sqrt[1 - Csc[c + d*x]]))   Subst[Int[(1
 + b*(x/a))^(n - 1/2)/(x*Sqrt[1 - b*(x/a)]), x], x, Csc[c + d*x]], x] /; Fr
eeQ[{a, b, c, d, n}, x] && EqQ[a^2 - b^2, 0] &&  !IntegerQ[2*n] && GtQ[a, 0
]
```

rule 4266
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[a^IntPar
t[n]*((a + b*Csc[c + d*x])^FracPart[n]/(1 + (b/a)*Csc[c + d*x])^FracPart[n]
)   Int[(1 + (b/a)*Csc[c + d*x])^n, x], x] /; FreeQ[{a, b, c, d, n}, x] &&
EqQ[a^2 - b^2, 0] &&  !IntegerQ[2*n] &&  !GtQ[a, 0]
```

3.33.4 Maple [F]

$$\int (a + a\csc(fx + e))^m \, dx$$

input
```
int((a+a*csc(f*x+e))^m,x)
```

output
```
int((a+a*csc(f*x+e))^m,x)
```

3.33.5 Fricas [F]

$$\int (a + a\csc(e + fx))^m \, dx = \int (a\csc(fx + e) + a)^m \, dx$$

input
```
integrate((a+a*csc(f*x+e))^m,x, algorithm="fricas")
```

output
```
integral((a*csc(f*x + e) + a)^m, x)
```

3.33.6 Sympy [F]

$$\int (a + a\csc(e + fx))^m \, dx = \int (a\csc(e + fx) + a)^m \, dx$$

input
```
integrate((a+a*csc(f*x+e))**m,x)
```

output
```
Integral((a*csc(e + f*x) + a)**m, x)
```

3.33.7 Maxima [F]

$$\int (a + a\csc(e + fx))^m \, dx = \int (a\csc(fx + e) + a)^m \, dx$$

input
```
integrate((a+a*csc(f*x+e))^m,x, algorithm="maxima")
```

output
```
integrate((a*csc(f*x + e) + a)^m, x)
```

3.33.8 Giac [F]

$$\int (a+a\csc(e+fx))^m\,dx=\int (a\csc(fx+e)+a)^m\,dx$$

input
```
integrate((a+a*csc(f*x+e))^m,x, algorithm="giac")
```

output
```
integrate((a*csc(f*x + e) + a)^m, x)
```

3.33.9 Mupad [F(-1)]

Timed out.

$$\int (a+a\csc(e+fx))^m\,dx=\int \left(a+\frac{a}{\sin(e+f\,x)}\right)^m\,dx$$

input
```
int((a + a/sin(e + f*x))^m,x)
```

output
```
int((a + a/sin(e + f*x))^m, x)
```

3.34 $\int (a+a\csc(e+fx))^m \sin(e+fx)\,dx$

3.34.1 Optimal result

Integrand size = 19, antiderivative size = 83

$$\int (a+a\csc(e+fx))^m \sin(e+fx)\,dx = \frac{\sqrt{2}\,\text{AppellF1}\left(\frac{1}{2}+m,\frac{1}{2},2,\frac{3}{2}+m,\frac{1}{2}(1+\csc(e+fx)),1+\csc(e+fx)\right)\cot(e+fx)(a+a\csc(e+fx))^m}{f(1+2m)\sqrt{1-\csc(e+fx)}}$$

output
```
AppellF1(1/2+m,2,1/2,3/2+m,1+csc(f*x+e),1/2+1/2*csc(f*x+e))*cot(f*x+e)*(a+
a*csc(f*x+e))^m*2^(1/2)/f/(1+2*m)/(1-csc(f*x+e))^(1/2)
```

3.34.2 Mathematica [F]

$$\int (a+a\csc(e+fx))^m \sin(e+fx)\,dx = \int (a+a\csc(e+fx))^m \sin(e+fx)\,dx$$

input
```
Integrate[(a + a*Csc[e + f*x])^m*Sin[e + f*x],x]
```

output
```
Integrate[(a + a*Csc[e + f*x])^m*Sin[e + f*x], x]
```

3.34.3 Rubi [A] (verified)

Time = 0.33 (sec) , antiderivative size = 83, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.263, Rules used = {3042, 4315, 3042, 4314, 153}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sin(e+fx)(a\csc(e+fx)+a)^m\,dx$$

$$\Big\downarrow \; 3042$$

$$\int \frac{(a\csc(e+fx)+a)^m}{\csc(e+fx)}dx$$

$$\Big\downarrow \; 4315$$

$$(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int (\csc(e+fx)+1)^m\sin(e+fx)dx$$

$$\Big\downarrow \; 3042$$

$$(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int \frac{(\csc(e+fx)+1)^m}{\csc(e+fx)}dx$$

$$\Big\downarrow \; 4314$$

$$\frac{\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m\int \frac{(\csc(e+fx)+1)^{m-\frac{1}{2}}\sin^2(e+fx)}{\sqrt{1-\csc(e+fx)}}d\csc(e+fx)}{f\sqrt{1-\csc(e+fx)}}$$

$$\Big\downarrow \; 153$$

$$\frac{\sqrt{2}\cot(e+fx)(a\csc(e+fx)+a)^m\operatorname{AppellF1}\left(m+\frac{1}{2},\frac{1}{2},2,m+\frac{3}{2},\frac{1}{2}(\csc(e+fx)+1),\csc(e+fx)+1\right)}{f(2m+1)\sqrt{1-\csc(e+fx)}}$$

input
```
Int[(a + a*Csc[e + f*x])^m*Sin[e + f*x],x]
```

output
```
(Sqrt[2]*AppellF1[1/2 + m, 1/2, 2, 3/2 + m, (1 + Csc[e + f*x])/2, 1 + Csc[
e + f*x]]*Cot[e + f*x]*(a + a*Csc[e + f*x])^m)/(f*(1 + 2*m)*Sqrt[1 - Csc[e
 + f*x]])
```

3.34.3.1 Defintions of rubi rules used

rule 153
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[(b*e - a*f)^p*((a + b*x)^(m + 1)/(b^(p + 1)*(m + 1)*Simp
lify[b/(b*c - a*d)]^n))*AppellF1[m + 1, -n, -p, m + 2, (-d)*((a + b*x)/(b*c
 - a*d)), (-f)*((a + b*x)/(b*e - a*f))], x] /; FreeQ[{a, b, c, d, e, f, m,
n}, x] &&  !IntegerQ[m] &&  !IntegerQ[n] && IntegerQ[p] && GtQ[Simplify[b/(
b*c - a*d)], 0] &&  !(GtQ[Simplify[d/(d*a - c*b)], 0] && SimplerQ[c + d*x,
a + b*x])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4314
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x
]]*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)*((a + b*x)^(m - 1/2
)/Sqrt[a - b*x]), x], x, Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, m, n},
x] && EqQ[a^2 - b^2, 0] &&  !IntegerQ[m] && GtQ[a, 0]
```

rule 4315
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Simp[a^IntPart[m]*((a + b*Csc[e + f*x])^FracPart[m
]/(1 + (b/a)*Csc[e + f*x])^FracPart[m])   Int[(1 + (b/a)*Csc[e + f*x])^m*(d
*Csc[e + f*x])^n, x], x] /; FreeQ[{a, b, d, e, f, m, n}, x] && EqQ[a^2 - b^
2, 0] &&  !IntegerQ[m] &&  !GtQ[a, 0]
```

3.34.4 Maple [F]

$$\int (a + a\csc(fx + e))^m \sin(fx + e) \, dx$$

input
```
int((a+a*csc(f*x+e))^m*sin(f*x+e),x)
```

output
```
int((a+a*csc(f*x+e))^m*sin(f*x+e),x)
```

3.34.5 Fricas [F]

$$\int (a+a\csc(e+fx))^m\sin(e+fx)\,dx=\int (a\csc\left(fx+e\right)+a)^m\sin\left(fx+e\right)\,dx$$

input
```
integrate((a+a*csc(f*x+e))^m*sin(f*x+e),x, algorithm="fricas")
```

output
```
integral((a*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

3.34.6 Sympy [F]

$$\int (a+a\csc(e+fx))^m\sin(e+fx)\,dx=\int (a(\csc\left(e+fx\right)+1))^m\sin\left(e+fx\right)dx$$

input
```
integrate((a+a*csc(f*x+e))**m*sin(f*x+e),x)
```

output
```
Integral((a*(csc(e + f*x) + 1))**m*sin(e + f*x), x)
```

3.34.7 Maxima [F]

$$\int (a+a\csc(e+fx))^m\sin(e+fx)\,dx=\int (a\csc\left(fx+e\right)+a)^m\sin\left(fx+e\right)\,dx$$

input
```
integrate((a+a*csc(f*x+e))^m*sin(f*x+e),x, algorithm="maxima")
```

output
```
integrate((a*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

3.34.8 Giac [F]

$$\int (a+a\csc(e+fx))^m\sin(e+fx)\,dx=\int (a\csc(fx+e)+a)^m\sin(fx+e)\,dx$$

input
```
integrate((a+a*csc(f*x+e))^m*sin(f*x+e),x, algorithm="giac")
```

output
```
integrate((a*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

3.34.9 Mupad [F(-1)]

Timed out.

$$\int (a+a\csc(e+fx))^m\sin(e+fx)\,dx=\int \sin(e+f\,x)\,\left(a+\frac{a}{\sin(e+f\,x)}\right)^m\,dx$$

input
```
int(sin(e + f*x)*(a + a/sin(e + f*x))^m,x)
```

output
```
int(sin(e + f*x)*(a + a/sin(e + f*x))^m, x)
```

3.35 $\int (a+a\csc(e+fx))^m \sin^2(e+fx)\,dx$

3.35.1 Optimal result

Integrand size = 21, antiderivative size = 84

$$\int (a+a\csc(e+fx))^m \sin^2(e+fx)\,dx = -\frac{\sqrt{2}\,\text{AppellF1}\left(\frac{1}{2}+m,\frac{1}{2},3,\frac{3}{2}+m,\frac{1}{2}(1+\csc(e+fx)),1+\csc(e+fx)\right)\cot(e+fx)(a+a\csc(e+fx))^m}{f(1+2m)\sqrt{1-\csc(e+fx)}}$$

output
```
-AppellF1(1/2+m,3,1/2,3/2+m,1+csc(f*x+e),1/2+1/2*csc(f*x+e))*cot(f*x+e)*(a
+a*csc(f*x+e))^m*2^(1/2)/f/(1+2*m)/(1-csc(f*x+e))^(1/2)
```

3.35.2 Mathematica [F]

$$\int (a+a\csc(e+fx))^m \sin^2(e+fx)\,dx = \int (a+a\csc(e+fx))^m \sin^2(e+fx)\,dx$$

input
```
Integrate[(a + a*Csc[e + f*x])^m*Sin[e + f*x]^2,x]
```

output
```
Integrate[(a + a*Csc[e + f*x])^m*Sin[e + f*x]^2, x]
```

3.35.3 Rubi [A] (verified)

Time = 0.33 (sec) , antiderivative size = 84, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.238, Rules used = {3042, 4315, 3042, 4314, 153}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sin^2(e+fx)(a\csc(e+fx)+a)^m\,dx$$

$$\Big\downarrow \text{ 3042}$$

$$\int \frac{(a\csc(e+fx)+a)^m}{\csc(e+fx)^2}dx$$

$$\Big\downarrow \text{ 4315}$$

$$(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int(\csc(e+fx)+1)^m\sin^2(e+fx)dx$$

$$\Big\downarrow \text{ 3042}$$

$$(\csc(e+fx)+1)^{-m}(a\csc(e+fx)+a)^m\int\frac{(\csc(e+fx)+1)^m}{\csc(e+fx)^2}dx$$

$$\Big\downarrow \text{ 4314}$$

$$\frac{\cot(e+fx)(\csc(e+fx)+1)^{-m-\frac{1}{2}}(a\csc(e+fx)+a)^m\int\frac{(\csc(e+fx)+1)^{m-\frac{1}{2}}\sin^3(e+fx)}{\sqrt{1-\csc(e+fx)}}d\csc(e+fx)}{f\sqrt{1-\csc(e+fx)}}$$

$$\Big\downarrow \text{ 153}$$

$$-\frac{\sqrt{2}\cot(e+fx)(a\csc(e+fx)+a)^m\operatorname{AppellF1}\left(m+\frac{1}{2},\frac{1}{2},3,m+\frac{3}{2},\frac{1}{2}(\csc(e+fx)+1),\csc(e+fx)+1\right)}{f(2m+1)\sqrt{1-\csc(e+fx)}}$$

input
```
Int[(a + a*Csc[e + f*x])^m*Sin[e + f*x]^2,x]
```

output
```
-((Sqrt[2]*AppellF1[1/2 + m, 1/2, 3, 3/2 + m, (1 + Csc[e + f*x])/2, 1 + Cs
c[e + f*x]]*Cot[e + f*x]*(a + a*Csc[e + f*x])^m)/(f*(1 + 2*m)*Sqrt[1 - Csc
[e + f*x]]))
```

3.35.3.1 Defintions of rubi rules used

rule 153
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[(b*e - a*f)^p*((a + b*x)^(m + 1)/(b^(p + 1)*(m + 1)*Simp
lify[b/(b*c - a*d)]^n))*AppellF1[m + 1, -n, -p, m + 2, (-d)*((a + b*x)/(b*c
 - a*d)), (-f)*((a + b*x)/(b*e - a*f))], x] /; FreeQ[{a, b, c, d, e, f, m,
n}, x] &&  !IntegerQ[m] &&  !IntegerQ[n] && IntegerQ[p] && GtQ[Simplify[b/(
b*c - a*d)], 0] &&  !(GtQ[Simplify[d/(d*a - c*b)], 0] && SimplerQ[c + d*x,
a + b*x])
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4314
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Simp[a^2*d*(Cot[e + f*x]/(f*Sqrt[a + b*Csc[e + f*x
]]*Sqrt[a - b*Csc[e + f*x]]))   Subst[Int[(d*x)^(n - 1)*((a + b*x)^(m - 1/2
)/Sqrt[a - b*x]), x], x, Csc[e + f*x]], x] /; FreeQ[{a, b, d, e, f, m, n},
x] && EqQ[a^2 - b^2, 0] &&  !IntegerQ[m] && GtQ[a, 0]
```

rule 4315
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_), x_Symbol] :> Simp[a^IntPart[m]*((a + b*Csc[e + f*x])^FracPart[m
]/(1 + (b/a)*Csc[e + f*x])^FracPart[m])   Int[(1 + (b/a)*Csc[e + f*x])^m*(d
*Csc[e + f*x])^n, x], x] /; FreeQ[{a, b, d, e, f, m, n}, x] && EqQ[a^2 - b^
2, 0] &&  !IntegerQ[m] &&  !GtQ[a, 0]
```

3.35.4 Maple [F]

$$\int (a+a\csc(fx+e))^m \sin(fx+e)^2\, dx$$

input
```
int((a+a*csc(f*x+e))^m*sin(f*x+e)^2,x)
```

output
```
int((a+a*csc(f*x+e))^m*sin(f*x+e)^2,x)
```

3.35.5 Fricas [F]

$$\int (a+a\csc(e+fx))^m\sin^2(e+fx)\,dx=\int (a\csc(fx+e)+a)^m\sin(fx+e)^2\,dx$$

input
```
integrate((a+a*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="fricas")
```

output
```
integral(-(cos(f*x + e)^2 - 1)*(a*csc(f*x + e) + a)^m, x)
```

3.35.6 Sympy [F]

$$\int (a+a\csc(e+fx))^m\sin^2(e+fx)\,dx=\int (a(\csc(e+fx)+1))^m\sin^2(e+fx)\,dx$$

input
```
integrate((a+a*csc(f*x+e))**m*sin(f*x+e)**2,x)
```

output
```
Integral((a*(csc(e + f*x) + 1))**m*sin(e + f*x)**2, x)
```

3.35.7 Maxima [F]

$$\int (a+a\csc(e+fx))^m\sin^2(e+fx)\,dx=\int (a\csc(fx+e)+a)^m\sin(fx+e)^2\,dx$$

input
```
integrate((a+a*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="maxima")
```

output
```
integrate((a*csc(f*x + e) + a)^m*sin(f*x + e)^2, x)
```

3.35.8 Giac [F]

$$\int (a + a\csc(e + fx))^m \sin^2(e + fx)\, dx = \int (a\csc(fx + e) + a)^m \sin(fx + e)^2\, dx$$

input
```
integrate((a+a*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="giac")
```

output
```
integrate((a*csc(f*x + e) + a)^m*sin(f*x + e)^2, x)
```

3.35.9 Mupad [F(-1)]

Timed out.

$$\int (a + a\csc(e + fx))^m \sin^2(e + fx)\, dx = \int \sin(e + f\,x)^2 \left(a + \frac{a}{\sin(e + f\,x)}\right)^m dx$$

input
```
int(sin(e + f*x)^2*(a + a/sin(e + f*x))^m,x)
```

output
```
int(sin(e + f*x)^2*(a + a/sin(e + f*x))^m, x)
```

3.36 $\int (a+b\csc(c+dx))^4\,dx$

3.36.1 Optimal result

Integrand size = 12, antiderivative size = 107

$$\int (a+b\csc(c+dx))^4\,dx = a^4x-\frac{2ab(2a^2+b^2)\operatorname{arctanh}(\cos(c+dx))}{d}-\frac{b^2(17a^2+2b^2)\cot(c+dx)}{3d}-\frac{4ab^3\cot(c+dx)\csc(c+dx)}{3d}-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))^2}{3d}$$

output
```
a^4*x-2*a*b*(2*a^2+b^2)*arctanh(cos(d*x+c))/d-1/3*b^2*(17*a^2+2*b^2)*cot(d
*x+c)/d-4/3*a*b^3*cot(d*x+c)*csc(d*x+c)/d-1/3*b^2*cot(d*x+c)*(a+b*csc(d*x+
c))^2/d
```

3.36.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 568 vs. $2(107) = 214$.

Time = 12.92 (sec) , antiderivative size = 568, normalized size of antiderivative = 5.31

$$\begin{aligned}\int (a+b\csc(c+dx))^4\,dx &= \frac{a^4(c+dx)(a+b\csc(c+dx))^4\sin^4(c+dx)}{d(b+a\sin(c+dx))^4} \\ &+ \frac{\left(-9a^2b^2\cos\left(\frac{1}{2}(c+dx)\right)-b^4\cos\left(\frac{1}{2}(c+dx)\right)\right)\csc\left(\frac{1}{2}(c+dx)\right)(a+b\csc(c+dx))^4\sin^4(c+dx)}{3d(b+a\sin(c+dx))^4} \\ &- \frac{ab^3\csc^2\left(\frac{1}{2}(c+dx)\right)(a+b\csc(c+dx))^4\sin^4(c+dx)}{2d(b+a\sin(c+dx))^4} \\ &- \frac{b^4\cot\left(\frac{1}{2}(c+dx)\right)\csc^2\left(\frac{1}{2}(c+dx)\right)(a+b\csc(c+dx))^4\sin^4(c+dx)}{24d(b+a\sin(c+dx))^4} \\ &- \frac{2(2a^3b+ab^3)(a+b\csc(c+dx))^4\log\left(\cos\left(\frac{1}{2}(c+dx)\right)\right)\sin^4(c+dx)}{d(b+a\sin(c+dx))^4} \\ &+ \frac{2(2a^3b+ab^3)(a+b\csc(c+dx))^4\log\left(\sin\left(\frac{1}{2}(c+dx)\right)\right)\sin^4(c+dx)}{d(b+a\sin(c+dx))^4} \\ &+ \frac{ab^3(a+b\csc(c+dx))^4\sec^2\left(\frac{1}{2}(c+dx)\right)\sin^4(c+dx)}{2d(b+a\sin(c+dx))^4} \\ &+ \frac{(a+b\csc(c+dx))^4\sec\left(\frac{1}{2}(c+dx)\right)\left(9a^2b^2\sin\left(\frac{1}{2}(c+dx)\right)+b^4\sin\left(\frac{1}{2}(c+dx)\right)\right)\sin^4(c+dx)}{3d(b+a\sin(c+dx))^4} \\ &+ \frac{b^4(a+b\csc(c+dx))^4\sec^2\left(\frac{1}{2}(c+dx)\right)\sin^4(c+dx)\tan\left(\frac{1}{2}(c+dx)\right)}{24d(b+a\sin(c+dx))^4}\end{aligned}$$

input
```
Integrate[(a + b*Csc[c + d*x])^4,x]
```

output
```
(a^4*(c + d*x)*(a + b*Csc[c + d*x])^4*Sin[c + d*x]^4)/(d*(b + a*Sin[c + d*
x])^4) + ((-9*a^2*b^2*Cos[(c + d*x)/2] - b^4*Cos[(c + d*x)/2])*Csc[(c + d*
x)/2]*(a + b*Csc[c + d*x])^4*Sin[c + d*x]^4)/(3*d*(b + a*Sin[c + d*x])^4)
- (a*b^3*Csc[(c + d*x)/2]^2*(a + b*Csc[c + d*x])^4*Sin[c + d*x]^4)/(2*d*(b
 + a*Sin[c + d*x])^4) - (b^4*Cot[(c + d*x)/2]*Csc[(c + d*x)/2]^2*(a + b*Cs
c[c + d*x])^4*Sin[c + d*x]^4)/(24*d*(b + a*Sin[c + d*x])^4) - (2*(2*a^3*b
+ a*b^3)*(a + b*Csc[c + d*x])^4*Log[Cos[(c + d*x)/2]]*Sin[c + d*x]^4)/(d*(
b + a*Sin[c + d*x])^4) + (2*(2*a^3*b + a*b^3)*(a + b*Csc[c + d*x])^4*Log[S
in[(c + d*x)/2]]*Sin[c + d*x]^4)/(d*(b + a*Sin[c + d*x])^4) + (a*b^3*(a +
b*Csc[c + d*x])^4*Sec[(c + d*x)/2]^2*Sin[c + d*x]^4)/(2*d*(b + a*Sin[c + d
*x])^4) + ((a + b*Csc[c + d*x])^4*Sec[(c + d*x)/2]*(9*a^2*b^2*Sin[(c + d*x
)/2] + b^4*Sin[(c + d*x)/2])*Sin[c + d*x]^4)/(3*d*(b + a*Sin[c + d*x])^4)
+ (b^4*(a + b*Csc[c + d*x])^4*Sec[(c + d*x)/2]^2*Sin[c + d*x]^4*Tan[(c + d
*x)/2])/(24*d*(b + a*Sin[c + d*x])^4)
```

3.36.3 Rubi [A] (verified)

Time = 0.40 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.07, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.417, Rules used = {3042, 4269, 3042, 4536, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (a+b\csc(c+dx))^4\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int (a+b\csc(c+dx))^4dx$$

$$\Big\downarrow \text{4269}$$

$$\frac{1}{3}\int (a+b\csc(c+dx))\left(3a^3+8b^2\csc^2(c+dx)a+b\left(9a^2+2b^2\right)\csc(c+dx)\right)dx-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))^2}{3d}$$

$$\Big\downarrow \text{3042}$$

$$\frac{1}{3}\int (a+b\csc(c+dx))\left(3a^3+8b^2\csc(c+dx)^2a+b\left(9a^2+2b^2\right)\csc(c+dx)\right)dx-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))^2}{3d}$$

$$\Big\downarrow \text{4536}$$

$$\frac{1}{3}\left(\frac{1}{2}\int \left(6a^4+12b\left(2a^2+b^2\right)\csc(c+dx)a+2b^2\left(17a^2+2b^2\right)\csc^2(c+dx)\right)dx-\frac{4ab^3\cot(c+dx)\csc(c+dx)}{d}\right)-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))^2}{3d}$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{3}\left(\frac{1}{2}\left(6a^4x-\frac{12ab\left(2a^2+b^2\right)\text{arctanh}(\cos(c+dx))}{d}-\frac{2b^2\left(17a^2+2b^2\right)\cot(c+dx)}{d}\right)-\frac{4ab^3\cot(c+dx)\csc(c+d}{d}\right.$$
$$\frac{b^2\cot(c+dx)(a+b\csc(c+dx))^2}{3d}$$

input
```
Int[(a + b*Csc[c + d*x])^4,x]
```

output
```
-1/3*(b^2*Cot[c + d*x]*(a + b*Csc[c + d*x])^2)/d + ((6*a^4*x - (12*a*b*(2*
a^2 + b^2)*ArcTanh[Cos[c + d*x]])/d - (2*b^2*(17*a^2 + 2*b^2)*Cot[c + d*x]
)/d)/2 - (4*a*b^3*Cot[c + d*x]*Csc[c + d*x])/d)/3
```

3.36.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4269
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(-b^2)*C
ot[c + d*x]*((a + b*Csc[c + d*x])^(n - 2)/(d*(n - 1))), x] + Simp[1/(n - 1)
   Int[(a + b*Csc[c + d*x])^(n - 3)*Simp[a^3*(n - 1) + (b*(b^2*(n - 2) + 3*
a^2*(n - 1)))*Csc[c + d*x] + (a*b^2*(3*n - 4))*Csc[c + d*x]^2, x], x], x] /
; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[n, 2] && IntegerQ[2*n]
```

rule 4536
```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbol] :> Simp[(-b)*C*Csc[e +
 f*x]*(Cot[e + f*x]/(2*f)), x] + Simp[1/2   Int[Simp[2*A*a + (2*B*a + b*(2*
A + C))*Csc[e + f*x] + 2*(a*C + B*b)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a
, b, e, f, A, B, C}, x]
```

3.36.4 Maple [A] (verified)

Time = 1.20 (sec) , antiderivative size = 112, normalized size of antiderivative = 1.05

method	result
derivativedivides	$\frac{a^4(dx+c)+4a^3b\ln(-\cot(dx+c)+\csc(dx+c))-6a^2b^2\cot(dx+c)+4a\,b^3\left(-\frac{\csc(dx+c)\cot(dx+c)}{2}+\frac{\ln(-\cot(dx+c)+\csc(dx+c))}{2}\right.}{d}$
default	$\frac{a^4(dx+c)+4a^3b\ln(-\cot(dx+c)+\csc(dx+c))-6a^2b^2\cot(dx+c)+4a\,b^3\left(-\frac{\csc(dx+c)\cot(dx+c)}{2}+\frac{\ln(-\cot(dx+c)+\csc(dx+c))}{2}\right.}{d}$
parts	$a^4x+\frac{b^4\left(-\frac{2}{3}-\frac{\csc(dx+c)^2}{3}\right)\cot(dx+c)}{d}-\frac{2a\,b^3\cot(dx+c)\csc(dx+c)}{d}+\frac{2a\,b^3\ln(-\cot(dx+c)+\csc(dx+c))}{d}-\frac{6a^2}{}$
parallelrisch	$\frac{\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^3b^4-\cot\left(\frac{dx}{2}+\frac{c}{2}\right)^3b^4+12\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2a\,b^3-12\cot\left(\frac{dx}{2}+\frac{c}{2}\right)^2a\,b^3+24a^4xd+72\tan\left(\frac{dx}{2}+\frac{c}{2}\right)a^2b^2+9\tan\left(\frac{dx}{2}\right.}{24d}$
norman	$\frac{a^4x\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^3-\frac{b^4}{24d}+\frac{b^4\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^6}{24d}-\frac{a\,b^3\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{2d}+\frac{a\,b^3\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^5}{2d}-\frac{3b^2\left(8a^2+b^2\right)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2}{8d}+\frac{3b^2\left(8a^2+b^2\right.}{}}{\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^3}$
risch	$a^4x+\frac{4b^2\left(-9ia^2\mathrm{e}^{4i(dx+c)}+3ab\,\mathrm{e}^{5i(dx+c)}+18ia^2\mathrm{e}^{2i(dx+c)}+3ib^2\mathrm{e}^{2i(dx+c)}-9ia^2-ib^2-3ab\,\mathrm{e}^{i(dx+c)}\right)}{3d\left(\mathrm{e}^{2i(dx+c)}-1\right)^3}-\frac{4a^3b\ln\left(\mathrm{e}^{i(dx}\right.}{d}$

input
```
int((a+b*csc(d*x+c))^4,x,method=_RETURNVERBOSE)
```

output
```
1/d*(a^4*(d*x+c)+4*a^3*b*ln(-cot(d*x+c)+csc(d*x+c))-6*a^2*b^2*cot(d*x+c)+4
*a*b^3*(-1/2*csc(d*x+c)*cot(d*x+c)+1/2*ln(-cot(d*x+c)+csc(d*x+c)))+b^4*(-2
/3-1/3*csc(d*x+c)^2)*cot(d*x+c))
```

3.36.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 217 vs. $2(101) = 202$.

Time = 0.25 (sec) , antiderivative size = 217, normalized size of antiderivative = 2.03

$$\int (a+b\csc(c+dx))^4\,dx =$$
$$-\frac{2\left(9\,a^2b^2+b^4\right)\cos\left(dx+c\right)^3-3\left(2\,a^3b+ab^3-\left(2\,a^3b+ab^3\right)\cos\left(dx+c\right)^2\right)\log\left(\frac{1}{2}\,\cos\left(dx+c\right)+\frac{1}{2}\right)\sin\left(\right.}{}$$

input
```
integrate((a+b*csc(d*x+c))^4,x, algorithm="fricas")
```

output
```
-1/3*(2*(9*a^2*b^2 + b^4)*cos(d*x + c)^3 - 3*(2*a^3*b + a*b^3 - (2*a^3*b +
 a*b^3)*cos(d*x + c)^2)*log(1/2*cos(d*x + c) + 1/2)*sin(d*x + c) + 3*(2*a^
3*b + a*b^3 - (2*a^3*b + a*b^3)*cos(d*x + c)^2)*log(-1/2*cos(d*x + c) + 1/
2)*sin(d*x + c) - 3*(6*a^2*b^2 + b^4)*cos(d*x + c) - 3*(a^4*d*x*cos(d*x +
c)^2 - a^4*d*x + 2*a*b^3*cos(d*x + c))*sin(d*x + c))/((d*cos(d*x + c)^2 -
d)*sin(d*x + c))
```

3.36.6 Sympy [F]

$$\int (a+b\csc(c+dx))^4\,dx = \int (a+b\csc{(c+dx)})^4\,dx$$

input
```
integrate((a+b*csc(d*x+c))**4,x)
```

output
```
Integral((a + b*csc(c + d*x))**4, x)
```

3.36.7 Maxima [A] (verification not implemented)

Time = 0.22 (sec) , antiderivative size = 125, normalized size of antiderivative = 1.17

$$\int (a+b\csc(c+dx))^4\,dx = a^4x + \frac{ab^3\left(\frac{2\cos(dx+c)}{\cos(dx+c)^2-1} - \log(\cos(dx+c)+1) + \log(\cos(dx+c)-1)\right)}{d} - \frac{4a^3b\log(\cot(dx+c)+\csc(dx+c))}{d} - \frac{6a^2b^2}{d\tan(dx+c)} - \frac{(3\tan(dx+c)^2+1)b^4}{3d\tan(dx+c)^3}$$

input
```
integrate((a+b*csc(d*x+c))^4,x, algorithm="maxima")
```

output
```
a^4*x + a*b^3*(2*cos(d*x + c)/(cos(d*x + c)^2 - 1) - log(cos(d*x + c) + 1)
 + log(cos(d*x + c) - 1))/d - 4*a^3*b*log(cot(d*x + c) + csc(d*x + c))/d -
 6*a^2*b^2/(d*tan(d*x + c)) - 1/3*(3*tan(d*x + c)^2 + 1)*b^4/(d*tan(d*x +
c)^3)
```

3.36.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 205 vs. $2(101) = 202$.

Time = 0.29 (sec) , antiderivative size = 205, normalized size of antiderivative = 1.92

$$\int (a+b\csc(c+dx))^4\,dx$$
$$= \frac{b^4\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^3+12\,ab^3\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^2+24\,(dx+c)a^4+72\,a^2b^2\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+9\,b^4\tan\left(\frac{1}{2}\,dx+\right.}{}$$

input
```
integrate((a+b*csc(d*x+c))^4,x, algorithm="giac")
```

output
```
1/24*(b^4*tan(1/2*d*x + 1/2*c)^3 + 12*a*b^3*tan(1/2*d*x + 1/2*c)^2 + 24*(d
*x + c)*a^4 + 72*a^2*b^2*tan(1/2*d*x + 1/2*c) + 9*b^4*tan(1/2*d*x + 1/2*c)
 + 48*(2*a^3*b + a*b^3)*log(abs(tan(1/2*d*x + 1/2*c))) - (176*a^3*b*tan(1/
2*d*x + 1/2*c)^3 + 88*a*b^3*tan(1/2*d*x + 1/2*c)^3 + 72*a^2*b^2*tan(1/2*d*
x + 1/2*c)^2 + 9*b^4*tan(1/2*d*x + 1/2*c)^2 + 12*a*b^3*tan(1/2*d*x + 1/2*c
) + b^4)/tan(1/2*d*x + 1/2*c)^3)/d
```

3.36.9 Mupad [B] (verification not implemented)

Time = 18.00 (sec) , antiderivative size = 314, normalized size of antiderivative = 2.93

$$\begin{aligned}\int (a+b\csc(c+dx))^4\,dx &= \frac{b^4\tan\left(\frac{c}{2}+\frac{dx}{2}\right)^3}{24\,d}-\frac{b^4\cot\left(\frac{c}{2}+\frac{dx}{2}\right)^3}{24\,d}\\&\quad-\frac{3\,b^4\cot\left(\frac{c}{2}+\frac{dx}{2}\right)}{8\,d}+\frac{3\,b^4\tan\left(\frac{c}{2}+\frac{dx}{2}\right)}{8\,d}\\&\quad+\frac{2\,a^4\,\mathrm{atan}\left(\frac{\cos\left(\frac{c}{2}+\frac{dx}{2}\right)a^3+4\,\sin\left(\frac{c}{2}+\frac{dx}{2}\right)a^2\,b+2\,\sin\left(\frac{c}{2}+\frac{dx}{2}\right)b^3}{-\sin\left(\frac{c}{2}+\frac{dx}{2}\right)a^3+4\,\cos\left(\frac{c}{2}+\frac{dx}{2}\right)a^2\,b+2\,\cos\left(\frac{c}{2}+\frac{dx}{2}\right)b^3}\right)}{d}\\&\quad+\frac{2\,a\,b^3\,\ln\left(\frac{\sin\left(\frac{c}{2}+\frac{dx}{2}\right)}{\cos\left(\frac{c}{2}+\frac{dx}{2}\right)}\right)}{d}+\frac{4\,a^3\,b\,\ln\left(\frac{\sin\left(\frac{c}{2}+\frac{dx}{2}\right)}{\cos\left(\frac{c}{2}+\frac{dx}{2}\right)}\right)}{d}\\&\quad-\frac{3\,a^2\,b^2\cot\left(\frac{c}{2}+\frac{dx}{2}\right)}{d}-\frac{a\,b^3\cot\left(\frac{c}{2}+\frac{dx}{2}\right)^2}{2\,d}\\&\quad+\frac{3\,a^2\,b^2\tan\left(\frac{c}{2}+\frac{dx}{2}\right)}{d}+\frac{a\,b^3\tan\left(\frac{c}{2}+\frac{dx}{2}\right)^2}{2\,d}\end{aligned}$$

input
```
int((a + b/sin(c + d*x))^4,x)
```

output
```
(b^4*tan(c/2 + (d*x)/2)^3)/(24*d) - (b^4*cot(c/2 + (d*x)/2)^3)/(24*d) - (3
*b^4*cot(c/2 + (d*x)/2))/(8*d) + (3*b^4*tan(c/2 + (d*x)/2))/(8*d) + (2*a^4
*atan((a^3*cos(c/2 + (d*x)/2) + 2*b^3*sin(c/2 + (d*x)/2) + 4*a^2*b*sin(c/2
 + (d*x)/2))/(2*b^3*cos(c/2 + (d*x)/2) - a^3*sin(c/2 + (d*x)/2) + 4*a^2*b*
cos(c/2 + (d*x)/2))))/d + (2*a*b^3*log(sin(c/2 + (d*x)/2)/cos(c/2 + (d*x)/
2)))/d + (4*a^3*b*log(sin(c/2 + (d*x)/2)/cos(c/2 + (d*x)/2)))/d - (3*a^2*b
^2*cot(c/2 + (d*x)/2))/d - (a*b^3*cot(c/2 + (d*x)/2)^2)/(2*d) + (3*a^2*b^2
*tan(c/2 + (d*x)/2))/d + (a*b^3*tan(c/2 + (d*x)/2)^2)/(2*d)
```

3.36. $\int (a+b\csc(c+dx))^4\,dx$

3.37 $\int (a+b\csc(c+dx))^3\,dx$

3.37.1 Optimal result

Integrand size = 12, antiderivative size = 73

$$\int (a+b\csc(c+dx))^3\,dx = a^3x-\frac{b(6a^2+b^2)\operatorname{arctanh}(\cos(c+dx))}{2d}-\frac{5ab^2\cot(c+dx)}{2d}-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))}{2d}$$

output
```
a^3*x-1/2*b*(6*a^2+b^2)*arctanh(cos(d*x+c))/d-5/2*a*b^2*cot(d*x+c)/d-1/2*b
^2*cot(d*x+c)*(a+b*csc(d*x+c))/d
```

3.37.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 152 vs. 2(73) = 146.

Time = 3.44 (sec) , antiderivative size = 152, normalized size of antiderivative = 2.08

$$\int (a+b\csc(c+dx))^3\,dx = \frac{8a^3c+8a^3dx-12ab^2\cot\left(\frac{1}{2}(c+dx)\right)-b^3\csc^2\left(\frac{1}{2}(c+dx)\right)-24a^2b\log\left(\cos\left(\frac{1}{2}(c+dx)\right)\right)-4b^3\log\left(\cos\left(\right.\right.}{}$$

input
```
Integrate[(a + b*Csc[c + d*x])^3,x]
```

output
```
(8*a^3*c + 8*a^3*d*x - 12*a*b^2*Cot[(c + d*x)/2] - b^3*Csc[(c + d*x)/2]^2
- 24*a^2*b*Log[Cos[(c + d*x)/2]] - 4*b^3*Log[Cos[(c + d*x)/2]] + 24*a^2*b*
Log[Sin[(c + d*x)/2]] + 4*b^3*Log[Sin[(c + d*x)/2]] + b^3*Sec[(c + d*x)/2]
^2 + 12*a*b^2*Tan[(c + d*x)/2])/(8*d)
```

3.37.3 Rubi [A] (verified)

Time = 0.25 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.03, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {3042, 4269, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (a+b\csc(c+dx))^3\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int (a+b\csc(c+dx))^3dx$$

$$\Big\downarrow \text{4269}$$

$$\frac{1}{2}\int \left(2a^3+5b^2\csc^2(c+dx)a+b\left(6a^2+b^2\right)\csc(c+dx)\right)dx-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))}{2d}$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{2}\left(2a^3x-\frac{b\left(6a^2+b^2\right)\text{arctanh}(\cos(c+dx))}{d}-\frac{5ab^2\cot(c+dx)}{d}\right)-\frac{b^2\cot(c+dx)(a+b\csc(c+dx))}{2d}$$

input
```
Int[(a + b*Csc[c + d*x])^3,x]
```

output
```
(2*a^3*x - (b*(6*a^2 + b^2)*ArcTanh[Cos[c + d*x]])/d - (5*a*b^2*Cot[c + d*
x])/d)/2 - (b^2*Cot[c + d*x]*(a + b*Csc[c + d*x]))/(2*d)
```

3.37.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4269
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[(-b^2)*C
ot[c + d*x]*((a + b*Csc[c + d*x])^(n - 2)/(d*(n - 1))), x] + Simp[1/(n - 1)
   Int[(a + b*Csc[c + d*x])^(n - 3)*Simp[a^3*(n - 1) + (b*(b^2*(n - 2) + 3*
a^2*(n - 1)))*Csc[c + d*x] + (a*b^2*(3*n - 4))*Csc[c + d*x]^2, x], x], x] /
; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[n, 2] && IntegerQ[2*n]
```

3.37.4 Maple [A] (verified)

Time = 0.72 (sec) , antiderivative size = 85, normalized size of antiderivative = 1.16

method	result
parts	$a^3x+\frac{b^3\left(-\frac{\csc(dx+c)\cot(dx+c)}{2}+\frac{\ln(-\cot(dx+c)+\csc(dx+c))}{2}\right)}{d}-\frac{3a^2b\ln(\csc(dx+c)+\cot(dx+c))}{d}-\frac{3a\,b^2\cot(dx+c)}{d}$
derivativedivides	$\frac{a^3(dx+c)+3a^2b\ln(-\cot(dx+c)+\csc(dx+c))-3a\,b^2\cot(dx+c)+b^3\left(-\frac{\csc(dx+c)\cot(dx+c)}{2}+\frac{\ln(-\cot(dx+c)+\csc(dx+c))}{2}\right)}{d}$
default	$\frac{a^3(dx+c)+3a^2b\ln(-\cot(dx+c)+\csc(dx+c))-3a\,b^2\cot(dx+c)+b^3\left(-\frac{\csc(dx+c)\cot(dx+c)}{2}+\frac{\ln(-\cot(dx+c)+\csc(dx+c))}{2}\right)}{d}$
parallelrisch	$\frac{4(6a^2b+b^3)\ln\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)+8a^3xd-\cot\left(\frac{dx}{2}+\frac{c}{2}\right)^2b^3+\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2b^3-12\cot\left(\frac{dx}{2}+\frac{c}{2}\right)a\,b^2+12\tan\left(\frac{dx}{2}+\frac{c}{2}\right)a\,b^2}{8d}$
norman	$\frac{a^3x\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2-\frac{b^3}{8d}+\frac{b^3\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^4}{8d}-\frac{3a\,b^2\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{2d}+\frac{3a\,b^2\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^3}{2d}}{\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2}+\frac{b(6a^2+b^2)\ln\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{2d}$
risch	$a^3x+\frac{b^2\left(-6ia\,e^{2i(dx+c)}+b\,e^{3i(dx+c)}+6ia+b\,e^{i(dx+c)}\right)}{d\left(e^{2i(dx+c)}-1\right)^2}-\frac{3b\ln\left(e^{i(dx+c)}+1\right)a^2}{d}-\frac{b^3\ln\left(e^{i(dx+c)}+1\right)}{2d}+\frac{3b\ln\left(e^{i(d}\right.}{}$

input
```
int((a+b*csc(d*x+c))^3,x,method=_RETURNVERBOSE)
```

output
```
a^3*x+b^3/d*(-1/2*csc(d*x+c)*cot(d*x+c)+1/2*ln(-cot(d*x+c)+csc(d*x+c)))-3*
a^2*b/d*ln(csc(d*x+c)+cot(d*x+c))-3*a*b^2*cot(d*x+c)/d
```

3.37.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 155 vs. $2(67) = 134$.

Time = 0.25 (sec) , antiderivative size = 155, normalized size of antiderivative = 2.12

$$\int (a+b\csc(c+dx))^3\,dx = \frac{4\,a^3dx\cos\left(dx+c\right)^2 - 4\,a^3dx + 12\,ab^2\cos\left(dx+c\right)\sin\left(dx+c\right) + 2\,b^3\cos\left(dx+c\right) + \left(6\,a^2b + b^3 - \left(6\,a^2b + \right.\right.}{4\left(d\cos\right.}$$

input
```
integrate((a+b*csc(d*x+c))^3,x, algorithm="fricas")
```

output
```
1/4*(4*a^3*d*x*cos(d*x + c)^2 - 4*a^3*d*x + 12*a*b^2*cos(d*x + c)*sin(d*x
+ c) + 2*b^3*cos(d*x + c) + (6*a^2*b + b^3 - (6*a^2*b + b^3)*cos(d*x + c)^
2)*log(1/2*cos(d*x + c) + 1/2) - (6*a^2*b + b^3 - (6*a^2*b + b^3)*cos(d*x
+ c)^2)*log(-1/2*cos(d*x + c) + 1/2))/(d*cos(d*x + c)^2 - d)
```

3.37.6 Sympy [F]

$$\int (a+b\csc(c+dx))^3\,dx = \int \left(a+b\csc\left(c+dx\right)\right)^3\,dx$$

input
```
integrate((a+b*csc(d*x+c))**3,x)
```

output
```
Integral((a + b*csc(c + d*x))**3, x)
```

3.37.7 Maxima [A] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 95, normalized size of antiderivative = 1.30

$$\int (a+b\csc(c+dx))^3\,dx = a^3x + \frac{b^3\left(\frac{2\cos(dx+c)}{\cos(dx+c)^2-1} - \log\left(\cos\left(dx+c\right)+1\right) + \log\left(\cos\left(dx+c\right)-1\right)\right)}{4\,d} - \frac{3\,a^2b\log\left(\cot\left(dx+c\right)+\csc\left(dx+c\right)\right)}{d} - \frac{3\,ab^2}{d\tan\left(dx+c\right)}$$

input
```
integrate((a+b*csc(d*x+c))^3,x, algorithm="maxima")
```

output
```
a^3*x + 1/4*b^3*(2*cos(d*x + c)/(cos(d*x + c)^2 - 1) - log(cos(d*x + c) +
1) + log(cos(d*x + c) - 1))/d - 3*a^2*b*log(cot(d*x + c) + csc(d*x + c))/d
 - 3*a*b^2/(d*tan(d*x + c))
```

3.37.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 134, normalized size of antiderivative = 1.84

$$\int (a+b\csc(c+dx))^3\,dx$$
$$= \frac{b^3\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^2+8\,(dx+c)a^3+12\,ab^2\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+4\,(6\,a^2b+b^3)\log\left(\left|\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)\right|\right)-\frac{36\,a}{}}{8\,d}$$

input
```
integrate((a+b*csc(d*x+c))^3,x, algorithm="giac")
```

output
```
1/8*(b^3*tan(1/2*d*x + 1/2*c)^2 + 8*(d*x + c)*a^3 + 12*a*b^2*tan(1/2*d*x +
 1/2*c) + 4*(6*a^2*b + b^3)*log(abs(tan(1/2*d*x + 1/2*c))) - (36*a^2*b*tan
(1/2*d*x + 1/2*c)^2 + 6*b^3*tan(1/2*d*x + 1/2*c)^2 + 12*a*b^2*tan(1/2*d*x
+ 1/2*c) + b^3)/tan(1/2*d*x + 1/2*c)^2)/d
```

3.37.9 Mupad [B] (verification not implemented)

Time = 18.59 (sec) , antiderivative size = 234, normalized size of antiderivative = 3.21

$$\int (a+b\csc(c+dx))^3\,dx = \frac{b^3\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)^2}{8\,d}-\frac{b^3\cot\left(\frac{c}{2}+\frac{d\,x}{2}\right)^2}{8\,d}+\frac{b^3\,\ln\left(\frac{\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)}{\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)}\right)}{2\,d}$$
$$+\frac{2\,a^3\,\mathrm{atan}\left(\frac{2\,\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a^3+6\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a^2\,b+\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,b^3}{-2\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a^3+6\,\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a^2\,b+\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,b^3}\right)}{d}$$
$$+\frac{3\,a^2\,b\,\ln\left(\frac{\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)}{\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)}\right)}{d}-\frac{3\,a\,b^2\cot\left(\frac{c}{2}+\frac{d\,x}{2}\right)}{2\,d}$$
$$+\frac{3\,a\,b^2\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)}{2\,d}$$

input
```
int((a + b/sin(c + d*x))^3,x)
```

output
```
(b^3*tan(c/2 + (d*x)/2)^2)/(8*d) - (b^3*cot(c/2 + (d*x)/2)^2)/(8*d) + (b^3
*log(sin(c/2 + (d*x)/2)/cos(c/2 + (d*x)/2)))/(2*d) + (2*a^3*atan((2*a^3*co
s(c/2 + (d*x)/2) + b^3*sin(c/2 + (d*x)/2) + 6*a^2*b*sin(c/2 + (d*x)/2))/(b
^3*cos(c/2 + (d*x)/2) - 2*a^3*sin(c/2 + (d*x)/2) + 6*a^2*b*cos(c/2 + (d*x)
/2))))/d + (3*a^2*b*log(sin(c/2 + (d*x)/2)/cos(c/2 + (d*x)/2)))/d - (3*a*b
^2*cot(c/2 + (d*x)/2))/(2*d) + (3*a*b^2*tan(c/2 + (d*x)/2))/(2*d)
```

3.37. $\int (a+b\csc(c+dx))^3\,dx$

3.38 $\int (a+b\csc(c+dx))^2\,dx$

3.38.1 Optimal result

Integrand size = 12, antiderivative size = 34

$$\int (a+b\csc(c+dx))^2\,dx=a^2x-\frac{2ab\text{arctanh}(\cos(c+dx))}{d}-\frac{b^2\cot(c+dx)}{d}$$

output
```
a^2*x-2*a*b*arctanh(cos(d*x+c))/d-b^2*cot(d*x+c)/d
```

3.38.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 76 vs. 2(34) = 68.

Time = 0.71 (sec) , antiderivative size = 76, normalized size of antiderivative = 2.24

$$\int (a+b\csc(c+dx))^2\,dx$$
$$=\frac{-b^2\cot\left(\frac{1}{2}(c+dx)\right)+2a\left(ac+adx-2b\log\left(\cos\left(\frac{1}{2}(c+dx)\right)\right)+2b\log\left(\sin\left(\frac{1}{2}(c+dx)\right)\right)\right)+b^2\tan\left(\frac{1}{2}(c+\right.}{2d}$$

input
```
Integrate[(a + b*Csc[c + d*x])^2,x]
```

output
```
(-(b^2*Cot[(c + d*x)/2]) + 2*a*(a*c + a*d*x - 2*b*Log[Cos[(c + d*x)/2]] +
2*b*Log[Sin[(c + d*x)/2]]) + b^2*Tan[(c + d*x)/2])/(2*d)
```

3.38.3 Rubi [A] (verified)

Time = 0.27 (sec) , antiderivative size = 34, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.500, Rules used = {3042, 4260, 3042, 4254, 24, 4257}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (a+b\csc(c+dx))^2\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int (a+b\csc(c+dx))^2dx$$

$$\Big\downarrow \text{4260}$$

$$2ab\int \csc(c+dx)dx+b^2\int \csc^2(c+dx)dx+a^2x$$

$$\Big\downarrow \text{3042}$$

$$2ab\int \csc(c+dx)dx+b^2\int \csc(c+dx)^2dx+a^2x$$

$$\Big\downarrow \text{4254}$$

$$2ab\int \csc(c+dx)dx-\frac{b^2\int 1d\cot(c+dx)}{d}+a^2x$$

$$\Big\downarrow \text{24}$$

$$2ab\int \csc(c+dx)dx+a^2x-\frac{b^2\cot(c+dx)}{d}$$

$$\Big\downarrow \text{4257}$$

$$a^2x-\frac{2ab\text{arctanh}(\cos(c+dx))}{d}-\frac{b^2\cot(c+dx)}{d}$$

input
```
Int[(a + b*Csc[c + d*x])^2,x]
```

output
```
a^2*x - (2*a*b*ArcTanh[Cos[c + d*x]])/d - (b^2*Cot[c + d*x])/d
```

3.38.3.1 Defintions of rubi rules used

rule 24
```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4254
```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> Simp[-d^(-1)   Subst[Int[Exp
andIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
 d}, x] && IGtQ[n/2, 0]
```

rule 4257
```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

rule 4260
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^2, x_Symbol] :> Simp[a^2*x, x] +
 (Simp[2*a*b   Int[Csc[c + d*x], x], x] + Simp[b^2   Int[Csc[c + d*x]^2, x]
, x]) /; FreeQ[{a, b, c, d}, x]
```

3.38.4 Maple [A] (verified)

Time = 0.65 (sec) , antiderivative size = 42, normalized size of antiderivative = 1.24

method	result	size
parts	$a^2x-\frac{b^2\cot(dx+c)}{d}-\frac{2ab\ln(\csc(dx+c)+\cot(dx+c))}{d}$	42
derivativedivides	$\frac{a^2(dx+c)+2ab\ln(-\cot(dx+c)+\csc(dx+c))-\cot(dx+c)b^2}{d}$	46
default	$\frac{a^2(dx+c)+2ab\ln(-\cot(dx+c)+\csc(dx+c))-\cot(dx+c)b^2}{d}$	46
parallelrisch	$\frac{2a^2xd+4\ln\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)ab+\tan\left(\frac{dx}{2}+\frac{c}{2}\right)b^2-\cot\left(\frac{dx}{2}+\frac{c}{2}\right)b^2}{2d}$	55
risch	$a^2x-\frac{2ib^2}{d\left(e^{2i(dx+c)}-1\right)}-\frac{2ab\ln\left(e^{i(dx+c)}+1\right)}{d}+\frac{2ab\ln\left(e^{i(dx+c)}-1\right)}{d}$	67
norman	$\frac{a^2x\tan\left(\frac{dx}{2}+\frac{c}{2}\right)-\frac{b^2}{2d}+\frac{b^2\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2}{2d}}{\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}+\frac{2ab\ln\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{d}$	73

input
```
int((a+b*csc(d*x+c))^2,x,method=_RETURNVERBOSE)
```

output
```
a^2*x-b^2*cot(d*x+c)/d-2*a*b/d*ln(csc(d*x+c)+cot(d*x+c))
```

3.38.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 77 vs. $2(34) = 68$.

Time = 0.25 (sec) , antiderivative size = 77, normalized size of antiderivative = 2.26

$$\int (a + b\csc(c + dx))^2 \, dx$$
$$= \frac{a^2dx\sin(dx + c) - ab\log\left(\frac{1}{2}\cos(dx + c) + \frac{1}{2}\right)\sin(dx + c) + ab\log\left(-\frac{1}{2}\cos(dx + c) + \frac{1}{2}\right)\sin(dx + c) - b}{d\sin(dx + c)}$$

input
```
integrate((a+b*csc(d*x+c))^2,x, algorithm="fricas")
```

output
```
(a^2*d*x*sin(d*x + c) - a*b*log(1/2*cos(d*x + c) + 1/2)*sin(d*x + c) + a*b
*log(-1/2*cos(d*x + c) + 1/2)*sin(d*x + c) - b^2*cos(d*x + c))/(d*sin(d*x
+ c))
```

3.38.6 Sympy [F]

$$\int (a + b\csc(c + dx))^2 \, dx = \int (a + b\csc(c + dx))^2 \, dx$$

input
```
integrate((a+b*csc(d*x+c))**2,x)
```

output
```
Integral((a + b*csc(c + d*x))**2, x)
```

3.38.7 Maxima [A] (verification not implemented)

Time = 0.22 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.26

$$\int (a + b\csc(c + dx))^2 \, dx = a^2x - \frac{2ab\log(\cot(dx + c) + \csc(dx + c))}{d} - \frac{b^2}{d\tan(dx + c)}$$

input
```
integrate((a+b*csc(d*x+c))^2,x, algorithm="maxima")
```

output
```
a^2*x - 2*a*b*log(cot(d*x + c) + csc(d*x + c))/d - b^2/(d*tan(d*x + c))
```

3.38.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 74 vs. $2(34) = 68$.

Time = 0.28 (sec) , antiderivative size = 74, normalized size of antiderivative = 2.18

$$\int (a + b\csc(c + dx))^2\,dx = \frac{2\,(dx+c)a^2 + 4\,ab\log\left(\left|\tan\left(\frac{1}{2}\,dx + \frac{1}{2}\,c\right)\right|\right) + b^2\tan\left(\frac{1}{2}\,dx + \frac{1}{2}\,c\right) - \frac{4\,ab\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+b^2}{\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)}}{2\,d}$$

input
```
integrate((a+b*csc(d*x+c))^2,x, algorithm="giac")
```

output
```
1/2*(2*(d*x + c)*a^2 + 4*a*b*log(abs(tan(1/2*d*x + 1/2*c))) + b^2*tan(1/2*
d*x + 1/2*c) - (4*a*b*tan(1/2*d*x + 1/2*c) + b^2)/tan(1/2*d*x + 1/2*c))/d
```

3.38.9 Mupad [B] (verification not implemented)

Time = 18.67 (sec) , antiderivative size = 105, normalized size of antiderivative = 3.09

$$\int (a + b\csc(c + dx))^2\,dx = \frac{2\,a^2\,\mathrm{atan}\left(\frac{a\,\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)+2\,b\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)}{2\,b\,\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)-a\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)}\right)}{d} - \frac{b^2\cot(c+d\,x)}{d} + \frac{2\,a\,b\,\ln\left(\frac{\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)}{\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)}\right)}{d}$$

input
```
int((a + b/sin(c + d*x))^2,x)
```

output
```
(2*a^2*atan((a*cos(c/2 + (d*x)/2) + 2*b*sin(c/2 + (d*x)/2))/(2*b*cos(c/2 +
 (d*x)/2) - a*sin(c/2 + (d*x)/2))))/d - (b^2*cot(c + d*x))/d + (2*a*b*log(
sin(c/2 + (d*x)/2)/cos(c/2 + (d*x)/2)))/d
```

3.39 $\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx$

3.39.1 Optimal result

Integrand size = 13, antiderivative size = 112

$$\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx = \frac{a(2a^2+b^2)\,\mathrm{arctanh}(\cos(x))}{2b^4} - \frac{2a^4\mathrm{arctanh}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b^4\sqrt{a^2-b^2}} - \frac{(3a^2+2b^2)\cot(x)}{3b^3} + \frac{a\cot(x)\csc(x)}{2b^2} - \frac{\cot(x)\csc^2(x)}{3b}$$

output
```
1/2*a*(2*a^2+b^2)*arctanh(cos(x))/b^4-1/3*(3*a^2+2*b^2)*cot(x)/b^3+1/2*a*c
ot(x)*csc(x)/b^2-1/3*cot(x)*csc(x)^2/b-2*a^4*arctanh((a+b*tan(1/2*x))/(a^2
-b^2)^(1/2))/b^4/(a^2-b^2)^(1/2)
```

3.39.2 Mathematica [A] (verified)

Time = 2.10 (sec) , antiderivative size = 125, normalized size of antiderivative = 1.12

$$\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx$$
$$= \frac{\frac{24a^4\arctan\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}} + b(3a^2+2b^2)\cos(3x)\csc^3(x) - 3b\cot(x)\csc(x)\left(-2ab+(a^2+2b^2)\csc(x)\right) + 6a(2a}{12b^4}$$

input
```
Integrate[Csc[x]^5/(a + b*Csc[x]),x]
```

output
```
((24*a^4*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqrt[-a^2 + b^2] + b*(
3*a^2 + 2*b^2)*Cos[3*x]*Csc[x]^3 - 3*b*Cot[x]*Csc[x]*(-2*a*b + (a^2 + 2*b^
2)*Csc[x]) + 6*a*(2*a^2 + b^2)*(Log[Cos[x/2]] - Log[Sin[x/2]]))/(12*b^4)
```

3.39.3 Rubi [A] (verified)

Time = 1.00 (sec) , antiderivative size = 139, normalized size of antiderivative = 1.24, number of steps used = 18, number of rules used = 17, $\frac{\text{number of rules}}{\text{integrand size}} = 1.308$, Rules used = {3042, 4338, 3042, 4580, 25, 3042, 4570, 27, 3042, 4486, 3042, 4257, 4318, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc^5(x)}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\csc(x)^5}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{4338}$$

$$\frac{\int \frac{\csc^2(x)\left(-3a\csc^2(x)+2b\csc(x)+2a\right)}{a+b\csc(x)}dx}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{\csc(x)^2\left(-3a\csc(x)^2+2b\csc(x)+2a\right)}{a+b\csc(x)}dx}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{4580}$$

$$\frac{\frac{\int -\frac{\csc(x)\left(3a^2-b\csc(x)a-2\left(3a^2+2b^2\right)\csc^2(x)\right)}{a+b\csc(x)}dx}{2b}+\frac{3a\cot(x)\csc(x)}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{25}$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\int \frac{\csc(x)\left(3a^2-b\csc(x)a-2\left(3a^2+2b^2\right)\csc^2(x)\right)}{a+b\csc(x)}dx}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\int \frac{\csc(x)\left(3a^2-b\csc(x)a-2\left(3a^2+2b^2\right)\csc(x)^2\right)}{a+b\csc(x)}dx}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{4570}$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\frac{\int \frac{3\csc(x)\left(ba^2+\left(2a^2+b^2\right)\csc(x)a\right)}{a+b\csc(x)}dx}{b}+\frac{2\left(3a^2+2b^2\right)\cot(x)}{b}}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{27}$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\frac{3\int \frac{\csc(x)\left(ba^2+\left(2a^2+b^2\right)\csc(x)a\right)}{a+b\csc(x)}dx}{b}+\frac{2\left(3a^2+2b^2\right)\cot(x)}{b}}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\frac{3\int \frac{\csc(x)\left(ba^2+\left(2a^2+b^2\right)\csc(x)a\right)}{a+b\csc(x)}dx}{b}+\frac{2\left(3a^2+2b^2\right)\cot(x)}{b}}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{4486}$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\frac{3\left(\frac{a\left(2a^2+b^2\right)\int\csc(x)dx}{b}-\frac{2a^4\int\frac{\csc(x)}{a+b\csc(x)}dx}{b}\right)}{b}+\frac{2\left(3a^2+2b^2\right)\cot(x)}{b}}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\frac{3\left(\frac{a\left(2a^2+b^2\right)\int\csc(x)dx}{b}-\frac{2a^4\int\frac{\csc(x)}{a+b\csc(x)}dx}{b}\right)}{b}+\frac{2\left(3a^2+2b^2\right)\cot(x)}{b}}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{4257}$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\frac{3\left(-\frac{2a^4\int\frac{\csc(x)}{a+b\csc(x)}dx}{b}-\frac{a\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}\right)}{b}+\frac{2\left(3a^2+2b^2\right)\cot(x)}{b}}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{4318}$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\frac{3\left(-\frac{2a^4\int\frac{1}{\frac{a\sin(x)}{b}+1}dx}{b^2}-\frac{a\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}\right)}{b}+\frac{2\left(3a^2+2b^2\right)\cot(x)}{b}}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\frac{3\left(-\frac{2a^4\int\frac{1}{\frac{a\sin(x)}{b}+1}dx}{b^2}-\frac{a\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}\right)}{b}+\frac{2\left(3a^2+2b^2\right)\cot(x)}{b}}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

3.39. $\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx$

$$\Big\downarrow \; 3139$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\frac{3\left(-\frac{4a^4\int \frac{1}{\tan^2\left(\frac{x}{2}\right)+\frac{2a\tan\left(\frac{x}{2}\right)}{b}+1}d\tan\left(\frac{x}{2}\right)}{b^2}-\frac{a\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}\right)}{b}+\frac{2\left(3a^2+2b^2\right)\cot(x)}{b}}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \; 1083$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\frac{3\left(\frac{8a^4\int \frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{b^2}-\frac{a\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}\right)}{b}+\frac{2\left(3a^2+2b^2\right)\cot(x)}{b}}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

$$\Big\downarrow \; 219$$

$$\frac{\frac{3a\cot(x)\csc(x)}{2b}-\frac{\frac{2\left(3a^2+2b^2\right)\cot(x)}{b}+\frac{3\left(\frac{4a^4\text{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{2\sqrt{a^2-b^2}}\right)}{b\sqrt{a^2-b^2}}-\frac{a\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}\right)}{b}}{2b}}{3b}-\frac{\cot(x)\csc^2(x)}{3b}$$

input
```
Int[Csc[x]^5/(a + b*Csc[x]),x]
```

output
```
-1/3*(Cot[x]*Csc[x]^2)/b + (-1/2*((3*(-((a*(2*a^2 + b^2)*ArcTanh[Cos[x]])/
b) + (4*a^4*ArcTanh[(b*((2*a)/b + 2*Tan[x/2]))/(2*Sqrt[a^2 - b^2])])/(b*Sq
rt[a^2 - b^2])))/b + (2*(3*a^2 + 2*b^2)*Cot[x])/b)/b + (3*a*Cot[x]*Csc[x])
/(2*b))/(3*b)
```

3.39. $\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx$

3.39.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4257
```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

rule 4318
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[1/b   Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

3.39. $\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx$

rule 4338
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(-d^3)*Cot[e + f*x]*((d*Csc[e + f*x])^(n - 3)/(b*f*
(n - 2))), x] + Simp[d^3/(b*(n - 2))   Int[(d*Csc[e + f*x])^(n - 3)*(Simp[a
*(n - 3) + b*(n - 3)*Csc[e + f*x] - a*(n - 2)*Csc[e + f*x]^2, x]/(a + b*Csc
[e + f*x])), x], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && Gt
Q[n, 3]
```

rule 4486
```
Int[(csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(B_.) + (A_)))/(csc[(
e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbol] :> Simp[B/b   Int[Csc[e + f*x],
 x], x] + Simp[(A*b - a*B)/b   Int[Csc[e + f*x]/(a + b*Csc[e + f*x]), x], x
] /; FreeQ[{a, b, e, f, A, B}, x] && NeQ[A*b - a*B, 0]
```

rule 4570
```
Int[csc[(e_.) + (f_.)*(x_)]*((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e
_.) + (f_.)*(x_)]^2*(C_.))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_S
ymbol] :> Simp[(-C)*Cot[e + f*x]*((a + b*Csc[e + f*x])^(m + 1)/(b*f*(m + 2)
)), x] + Simp[1/(b*(m + 2))   Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m*Simp[
b*A*(m + 2) + b*C*(m + 1) + (b*B*(m + 2) - a*C)*Csc[e + f*x], x], x], x] /;
 FreeQ[{a, b, e, f, A, B, C, m}, x] &&  !LtQ[m, -1]
```

rule 4580
```
Int[csc[(e_.) + (f_.)*(x_)]^2*((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[
(e_.) + (f_.)*(x_)]^2*(C_.))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x
_Symbol] :> Simp[(-C)*Csc[e + f*x]*Cot[e + f*x]*((a + b*Csc[e + f*x])^(m +
1)/(b*f*(m + 3))), x] + Simp[1/(b*(m + 3))   Int[Csc[e + f*x]*(a + b*Csc[e
+ f*x])^m*Simp[a*C + b*(C*(m + 2) + A*(m + 3))*Csc[e + f*x] - (2*a*C - b*B*
(m + 3))*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a, b, e, f, A, B, C, m}, x] &
& NeQ[a^2 - b^2, 0] &&  !LtQ[m, -1]
```

3.39.4 Maple [A] (verified)

Time = 0.75 (sec) , antiderivative size = 156, normalized size of antiderivative = 1.39

method	result
default	$\frac{\frac{\tan\left(\frac{x}{2}\right)^{3} b^{2}}{3}-a b \tan\left(\frac{x}{2}\right)^{2}+4 \tan\left(\frac{x}{2}\right) a^{2}+3 \tan\left(\frac{x}{2}\right) b^{2}}{8 b^{3}}+\frac{2 a^{4} \arctan\left(\frac{2 b \tan\left(\frac{x}{2}\right)+2 a}{2 \sqrt{-a^{2}+b^{2}}}\right)}{b^{4} \sqrt{-a^{2}+b^{2}}}-\frac{1}{24 b \tan\left(\frac{x}{2}\right)^{3}}-\frac{4 a^{2}+3 b^{2}}{8 b^{3} \tan\left(\frac{x}{2}\right)}+\frac{a}{8 b^{2} \tan\left(\frac{x}{2}\right)^{2}} \cdots$
risch	$\frac{i\left(3 i a b\, e^{5 i x}-6 a^{2} e^{4 i x}-3 i a b\, e^{i x}+12 a^{2} e^{2 i x}+12 b^{2} e^{2 i x}-6 a^{2}-4 b^{2}\right)}{3 b^{3} \left(e^{2 i x}-1\right)^{3}}-\frac{i a^{4} \ln\left(e^{i x}+\frac{i\left(\sqrt{-a^{2}+b^{2}}\, b+a^{2}-b^{2}\right)}{a \sqrt{-a^{2}+b^{2}}}\right)}{\sqrt{-a^{2}+b^{2}}\, b^{4}}+\frac{i a^{4} \ln\left(e^{i x}+\cdots\right)}{\sqrt{-a^{2}+b^{2}}\, b^{4}} \cdots$

input
```
int(csc(x)^5/(a+b*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
1/8/b^3*(1/3*tan(1/2*x)^3*b^2-a*b*tan(1/2*x)^2+4*tan(1/2*x)*a^2+3*tan(1/2*
x)*b^2)+2/b^4*a^4/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*x)+2*a)/(-a^2+b
^2)^(1/2))-1/24/b/tan(1/2*x)^3-1/8*(4*a^2+3*b^2)/b^3/tan(1/2*x)+1/8*a/b^2/
tan(1/2*x)^2-1/2/b^4*a*(2*a^2+b^2)*ln(tan(1/2*x))
```

3.39.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 280 vs. $2(98) = 196$.

Time = 0.35 (sec) , antiderivative size = 607, normalized size of antiderivative = 5.42

$$\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx$$
$$= \left[\frac{4\left(3\,a^4b-a^2b^3-2\,b^5\right)\cos\left(x\right)^3-6\left(a^4\cos\left(x\right)^2-a^4\right)\sqrt{a^2-b^2}\log\left(-\frac{(a^2-2\,b^2)\cos(x)^2+2\,ab\sin(x)+a^2+b^2-2\,(b\cos(}{a^2\cos(x)^2-2\,ab\sin(x)-a}\right.}{}\right.$$

input
```
integrate(csc(x)^5/(a+b*csc(x)),x, algorithm="fricas")
```

output
```
[1/12*(4*(3*a^4*b - a^2*b^3 - 2*b^5)*cos(x)^3 - 6*(a^4*cos(x)^2 - a^4)*sqr
t(a^2 - b^2)*log(-((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2 + b^2 - 2*(
b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2*a*b*sin(x)
- a^2 - b^2))*sin(x) + 6*(a^3*b^2 - a*b^4)*cos(x)*sin(x) + 3*(2*a^5 - a^3*
b^2 - a*b^4 - (2*a^5 - a^3*b^2 - a*b^4)*cos(x)^2)*log(1/2*cos(x) + 1/2)*si
n(x) - 3*(2*a^5 - a^3*b^2 - a*b^4 - (2*a^5 - a^3*b^2 - a*b^4)*cos(x)^2)*lo
g(-1/2*cos(x) + 1/2)*sin(x) - 12*(a^4*b - b^5)*cos(x))/((a^2*b^4 - b^6 - (
a^2*b^4 - b^6)*cos(x)^2)*sin(x)), 1/12*(4*(3*a^4*b - a^2*b^3 - 2*b^5)*cos(
x)^3 + 12*(a^4*cos(x)^2 - a^4)*sqrt(-a^2 + b^2)*arctan(-sqrt(-a^2 + b^2)*(
b*sin(x) + a)/((a^2 - b^2)*cos(x)))*sin(x) + 6*(a^3*b^2 - a*b^4)*cos(x)*si
n(x) + 3*(2*a^5 - a^3*b^2 - a*b^4 - (2*a^5 - a^3*b^2 - a*b^4)*cos(x)^2)*lo
g(1/2*cos(x) + 1/2)*sin(x) - 3*(2*a^5 - a^3*b^2 - a*b^4 - (2*a^5 - a^3*b^2
 - a*b^4)*cos(x)^2)*log(-1/2*cos(x) + 1/2)*sin(x) - 12*(a^4*b - b^5)*cos(x
))/((a^2*b^4 - b^6 - (a^2*b^4 - b^6)*cos(x)^2)*sin(x))]
```

3.39.6 Sympy [F]

$$\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx = \int \frac{\csc^5(x)}{a+b\csc(x)}\,dx$$

input
```
integrate(csc(x)**5/(a+b*csc(x)),x)
```

output
```
Integral(csc(x)**5/(a + b*csc(x)), x)
```

3.39.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(csc(x)^5/(a+b*csc(x)),x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.39.8 Giac [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 194, normalized size of antiderivative = 1.73

$$\begin{aligned}\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx &= \frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)a^4}{\sqrt{-a^2+b^2}\,b^4} \\ &+ \frac{b^2\tan\left(\frac{1}{2}x\right)^3-3\,ab\tan\left(\frac{1}{2}x\right)^2+12\,a^2\tan\left(\frac{1}{2}x\right)+9\,b^2\tan\left(\frac{1}{2}x\right)}{24\,b^3} \\ &- \frac{(2\,a^3+ab^2)\log\left(\left|\tan\left(\frac{1}{2}x\right)\right|\right)}{2\,b^4} \\ &+ \frac{44\,a^3\tan\left(\frac{1}{2}x\right)^3+22\,ab^2\tan\left(\frac{1}{2}x\right)^3-12\,a^2b\tan\left(\frac{1}{2}x\right)^2-9\,b^3\tan\left(\frac{1}{2}x\right)^2+3\,ab^2\tan\left(\frac{1}{2}x\right)-b^3}{24\,b^4\tan\left(\frac{1}{2}x\right)^3}\end{aligned}$$

input
```
integrate(csc(x)^5/(a+b*csc(x)),x, algorithm="giac")
```

output
```
2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
 b^2)))*a^4/(sqrt(-a^2 + b^2)*b^4) + 1/24*(b^2*tan(1/2*x)^3 - 3*a*b*tan(1/
2*x)^2 + 12*a^2*tan(1/2*x) + 9*b^2*tan(1/2*x))/b^3 - 1/2*(2*a^3 + a*b^2)*l
og(abs(tan(1/2*x)))/b^4 + 1/24*(44*a^3*tan(1/2*x)^3 + 22*a*b^2*tan(1/2*x)^
3 - 12*a^2*b*tan(1/2*x)^2 - 9*b^3*tan(1/2*x)^2 + 3*a*b^2*tan(1/2*x) - b^3)
/(b^4*tan(1/2*x)^3)
```

3.39.9 Mupad [B] (verification not implemented)

Time = 19.02 (sec) , antiderivative size = 588, normalized size of antiderivative = 5.25

$$\int \frac{\csc^5(x)}{a+b\csc(x)}\,dx =$$

$$-\frac{b^2\left(\frac{3\,a\,\sin(2\,x)\,\sqrt{a^2-b^2}}{4}+\frac{3\,a\,\sin(3\,x)\,\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)\sqrt{a^2-b^2}}{8}-\frac{9\,a\,\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)\sin(x)\,\sqrt{a^2-b^2}}{8}\right)+b^3\left(\frac{\cos(3\,x)\,\sqrt{a^2-b^2}}{2}-\frac{3\cos}{}\right.}{}$$

input
```
int(1/(sin(x)^5*(a + b/sin(x))),x)
```

output
```
-(b^2*((3*a*sin(2*x)*(a^2 - b^2)^(1/2))/4 + (3*a*sin(3*x)*log(sin(x/2)/cos
(x/2))*(a^2 - b^2)^(1/2))/8 - (9*a*log(sin(x/2)/cos(x/2))*sin(x)*(a^2 - b^
2)^(1/2))/8) + b^3*((cos(3*x)*(a^2 - b^2)^(1/2))/2 - (3*cos(x)*(a^2 - b^2)
^(1/2))/2) - b*((3*a^2*cos(x)*(a^2 - b^2)^(1/2))/4 - (3*a^2*cos(3*x)*(a^2
- b^2)^(1/2))/4) + (a^4*atan((a^4*sin(x/2)*(a^2 - b^2)^(1/2)*8i - b^4*sin(
x/2)*(a^2 - b^2)^(1/2)*1i + a*b^3*cos(x/2)*(a^2 - b^2)^(1/2)*1i + a^3*b*co
s(x/2)*(a^2 - b^2)^(1/2)*4i)/(b^5*cos(x/2) - 8*a^5*sin(x/2) + a^2*b^3*cos(
x/2) + 4*a^3*b^2*sin(x/2) - 4*a^4*b*cos(x/2) + 2*a*b^4*sin(x/2)))*sin(x)*9
i)/2 - (a^4*atan((a^4*sin(x/2)*(a^2 - b^2)^(1/2)*8i - b^4*sin(x/2)*(a^2 -
b^2)^(1/2)*1i + a*b^3*cos(x/2)*(a^2 - b^2)^(1/2)*1i + a^3*b*cos(x/2)*(a^2
- b^2)^(1/2)*4i)/(b^5*cos(x/2) - 8*a^5*sin(x/2) + a^2*b^3*cos(x/2) + 4*a^3
*b^2*sin(x/2) - 4*a^4*b*cos(x/2) + 2*a*b^4*sin(x/2)))*sin(3*x)*3i)/2 - (9*
a^3*log(sin(x/2)/cos(x/2))*sin(x)*(a^2 - b^2)^(1/2))/4 + (3*a^3*sin(3*x)*l
og(sin(x/2)/cos(x/2))*(a^2 - b^2)^(1/2))/4)/((3*b^4*sin(3*x)*(a^2 - b^2)^(
1/2))/4 - (9*b^4*sin(x)*(a^2 - b^2)^(1/2))/4)
```

3.40 $\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx$

3.40.1 Optimal result

Integrand size = 13, antiderivative size = 84

$$\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx = -\frac{(2a^2+b^2)\,\text{arctanh}(\cos(x))}{2b^3} + \frac{2a^3\text{arctanh}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b^3\sqrt{a^2-b^2}} + \frac{a\cot(x)}{b^2} - \frac{\cot(x)\csc(x)}{2b}$$

output
```
-1/2*(2*a^2+b^2)*arctanh(cos(x))/b^3+a*cot(x)/b^2-1/2*cot(x)*csc(x)/b+2*a^
3*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/b^3/(a^2-b^2)^(1/2)
```

3.40.2 Mathematica [A] (verified)

Time = 0.72 (sec) , antiderivative size = 144, normalized size of antiderivative = 1.71

$$\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx = \frac{-\frac{16a^3\arctan\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}} + 4ab\cot\left(\frac{x}{2}\right) - b^2\csc^2\left(\frac{x}{2}\right) - 8a^2\log\left(\cos\left(\frac{x}{2}\right)\right) - 4b^2\log\left(\cos\left(\frac{x}{2}\right)\right) + 8a^2\log\left(\sin\left(\frac{x}{2}\right)\right.}{8b^3}$$

input
```
Integrate[Csc[x]^4/(a + b*Csc[x]),x]
```

output
```
((-16*a^3*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqrt[-a^2 + b^2] + 4*
a*b*Cot[x/2] - b^2*Csc[x/2]^2 - 8*a^2*Log[Cos[x/2]] - 4*b^2*Log[Cos[x/2]]
+ 8*a^2*Log[Sin[x/2]] + 4*b^2*Log[Sin[x/2]] + b^2*Sec[x/2]^2 - 4*a*b*Tan[x
/2])/(8*b^3)
```

3.40.3 Rubi [A] (verified)

Time = 0.70 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.25, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.000, Rules used = {3042, 4338, 3042, 4570, 3042, 4486, 3042, 4257, 4318, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc^4(x)}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\csc(x)^4}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{4338}$$

$$\frac{\int \frac{\csc(x)\left(-2a\csc^2(x)+b\csc(x)+a\right)}{a+b\csc(x)}dx}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{\csc(x)\left(-2a\csc(x)^2+b\csc(x)+a\right)}{a+b\csc(x)}dx}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

$$\Big\downarrow \text{4570}$$

$$\frac{\frac{\int \frac{\csc(x)\left(ab+\left(2a^2+b^2\right)\csc(x)\right)}{a+b\csc(x)}dx}{b}+\frac{2a\cot(x)}{b}}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\frac{\int \frac{\csc(x)\left(ab+\left(2a^2+b^2\right)\csc(x)\right)}{a+b\csc(x)}dx}{b}+\frac{2a\cot(x)}{b}}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

$$\Big\downarrow \text{4486}$$

$$\frac{\frac{\frac{\left(2a^2+b^2\right)\int \csc(x)dx}{b}-\frac{2a^3\int \frac{\csc(x)}{a+b\csc(x)}dx}{b}}{b}+\frac{2a\cot(x)}{b}}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

$$\Big\downarrow \; 3042$$

$$\frac{\frac{\frac{\left(2a^2+b^2\right)\int \csc(x)dx}{b}-\frac{2a^3\int \frac{\csc(x)}{a+b\csc(x)}dx}{b}}{b}+\frac{2a\cot(x)}{b}}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

$$\Big\downarrow \; 4257$$

$$\frac{\frac{-\frac{2a^3\int \frac{\csc(x)}{a+b\csc(x)}dx}{b}-\frac{\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}}{b}+\frac{2a\cot(x)}{b}}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

$$\Big\downarrow \; 4318$$

$$\frac{\frac{-\frac{2a^3\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{b^2}-\frac{\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}}{b}+\frac{2a\cot(x)}{b}}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

$$\Big\downarrow \; 3042$$

$$\frac{\frac{-\frac{2a^3\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{b^2}-\frac{\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}}{b}+\frac{2a\cot(x)}{b}}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

$$\Big\downarrow \; 3139$$

$$\frac{\frac{-\frac{4a^3\int \frac{1}{\tan^2\left(\frac{x}{2}\right)+\frac{2a\tan\left(\frac{x}{2}\right)}{b}+1}d\tan\left(\frac{x}{2}\right)}{b^2}-\frac{\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}}{b}+\frac{2a\cot(x)}{b}}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

$$\Big\downarrow \; 1083$$

$$\frac{\frac{\frac{8a^3\int \frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{b^2}-\frac{\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}}{b}+\frac{2a\cot(x)}{b}}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

$$\Big\downarrow \; 219$$

$$\frac{\frac{\frac{4a^3\text{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{2\sqrt{a^2-b^2}}\right)}{b\sqrt{a^2-b^2}}-\frac{\left(2a^2+b^2\right)\text{arctanh}(\cos(x))}{b}}{b}+\frac{2a\cot(x)}{b}}{2b}-\frac{\cot(x)\csc(x)}{2b}$$

input
```
Int[Csc[x]^4/(a + b*Csc[x]),x]
```

output
```
((-(((2*a^2 + b^2)*ArcTanh[Cos[x]])/b) + (4*a^3*ArcTanh[(b*((2*a)/b + 2*Ta
n[x/2]))/(2*Sqrt[a^2 - b^2])])/(b*Sqrt[a^2 - b^2]))/b + (2*a*Cot[x])/b)/(2
*b) - (Cot[x]*Csc[x])/(2*b)
```

3.40. $\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx$

3.40.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4257
```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

rule 4318
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[1/b   Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

rule 4338
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[(-d^3)*Cot[e + f*x]*((d*Csc[e + f*x])^(n - 3)/(b*f*
(n - 2))), x] + Simp[d^3/(b*(n - 2))   Int[(d*Csc[e + f*x])^(n - 3)*(Simp[a
*(n - 3) + b*(n - 3)*Csc[e + f*x] - a*(n - 2)*Csc[e + f*x]^2, x]/(a + b*Csc
[e + f*x])), x], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && Gt
Q[n, 3]
```

rule 4486
```
Int[(csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(B_.) + (A_)))/(csc[(
e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbol] :> Simp[B/b   Int[Csc[e + f*x],
 x], x] + Simp[(A*b - a*B)/b   Int[Csc[e + f*x]/(a + b*Csc[e + f*x]), x], x
] /; FreeQ[{a, b, e, f, A, B}, x] && NeQ[A*b - a*B, 0]
```

3.40. $\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx$

rule 4570
```
Int[csc[(e_.) + (f_.)*(x_)]*((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e
_.) + (f_.)*(x_)]^2*(C_.))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_S
ymbol] :> Simp[(-C)*Cot[e + f*x]*((a + b*Csc[e + f*x])^(m + 1)/(b*f*(m + 2)
)), x] + Simp[1/(b*(m + 2))   Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m*Simp[
b*A*(m + 2) + b*C*(m + 1) + (b*B*(m + 2) - a*C)*Csc[e + f*x], x], x], x] /;
 FreeQ[{a, b, e, f, A, B, C, m}, x] &&  !LtQ[m, -1]
```

3.40.4 Maple [A] (verified)

Time = 0.57 (sec) , antiderivative size = 112, normalized size of antiderivative = 1.33

method	result
default	$-\frac{-\frac{b \tan\left(\frac{x}{2}\right)^2}{2}+2a\tan\left(\frac{x}{2}\right)}{4b^2}-\frac{1}{8b\tan\left(\frac{x}{2}\right)^2}+\frac{(4a^2+2b^2)\ln\left(\tan\left(\frac{x}{2}\right)\right)}{4b^3}+\frac{a}{2b^2\tan\left(\frac{x}{2}\right)}-\frac{2a^3\arctan\left(\frac{2b\tan\left(\frac{x}{2}\right)+2a}{2\sqrt{-a^2+b^2}}\right)}{b^3\sqrt{-a^2+b^2}}$
risch	$\frac{2ia\,e^{2ix}+b\,e^{3ix}-2ia+b\,e^{ix}}{(e^{2ix}-1)^2b^2}-\frac{\ln(e^{ix}+1)a^2}{b^3}-\frac{\ln(e^{ix}+1)}{2b}+\frac{a^3\ln\left(e^{ix}+\frac{ib\sqrt{a^2-b^2}+a^2-b^2}{\sqrt{a^2-b^2}\,a}\right)}{\sqrt{a^2-b^2}\,b^3}-\frac{a^3\ln\left(e^{ix}+\frac{ib\sqrt{a^2-b^2}-a^2+b^2}{\sqrt{a^2-b^2}\,a}\right)}{\sqrt{a^2-b^2}\,b^3}-$

input
```
int(csc(x)^4/(a+b*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
-1/4/b^2*(-1/2*b*tan(1/2*x)^2+2*a*tan(1/2*x))-1/8/b/tan(1/2*x)^2+1/4/b^3*(
4*a^2+2*b^2)*ln(tan(1/2*x))+1/2*a/b^2/tan(1/2*x)-2/b^3*a^3/(-a^2+b^2)^(1/2
)*arctan(1/2*(2*b*tan(1/2*x)+2*a)/(-a^2+b^2)^(1/2))
```

3.40.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 239 vs. 2(74) = 148.

Time = 0.36 (sec) , antiderivative size = 524, normalized size of antiderivative = 6.24

$$\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx$$
$$=\left[\frac{4\left(a^3b-ab^3\right)\cos(x)\sin(x)-2\left(a^3\cos(x)^2-a^3\right)\sqrt{a^2-b^2}\log\left(\frac{(a^2-2b^2)\cos(x)^2+2ab\sin(x)+a^2+b^2+2(b\cos(x)\sin(x)}{a^2\cos(x)^2-2ab\sin(x)-a^2-b^2}\right.}{}\right.$$

input
```
integrate(csc(x)^4/(a+b*csc(x)),x, algorithm="fricas")
```

output
```
[1/4*(4*(a^3*b - a*b^3)*cos(x)*sin(x) - 2*(a^3*cos(x)^2 - a^3)*sqrt(a^2 -
b^2)*log(((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2 + b^2 + 2*(b*cos(x)*
sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2*a*b*sin(x) - a^2 - b
^2)) - 2*(a^2*b^2 - b^4)*cos(x) - (2*a^4 - a^2*b^2 - b^4 - (2*a^4 - a^2*b^
2 - b^4)*cos(x)^2)*log(1/2*cos(x) + 1/2) + (2*a^4 - a^2*b^2 - b^4 - (2*a^4
 - a^2*b^2 - b^4)*cos(x)^2)*log(-1/2*cos(x) + 1/2))/(a^2*b^3 - b^5 - (a^2*
b^3 - b^5)*cos(x)^2), 1/4*(4*(a^3*b - a*b^3)*cos(x)*sin(x) - 4*(a^3*cos(x)
^2 - a^3)*sqrt(-a^2 + b^2)*arctan(-sqrt(-a^2 + b^2)*(b*sin(x) + a)/((a^2 -
 b^2)*cos(x))) - 2*(a^2*b^2 - b^4)*cos(x) - (2*a^4 - a^2*b^2 - b^4 - (2*a^
4 - a^2*b^2 - b^4)*cos(x)^2)*log(1/2*cos(x) + 1/2) + (2*a^4 - a^2*b^2 - b^
4 - (2*a^4 - a^2*b^2 - b^4)*cos(x)^2)*log(-1/2*cos(x) + 1/2))/(a^2*b^3 - b
^5 - (a^2*b^3 - b^5)*cos(x)^2)]
```

3.40.6 Sympy [F]

$$\int \frac{\csc^4(x)}{a + b\csc(x)}\,dx = \int \frac{\csc^4(x)}{a + b\csc(x)}\,dx$$

input
```
integrate(csc(x)**4/(a+b*csc(x)),x)
```

output
```
Integral(csc(x)**4/(a + b*csc(x)), x)
```

3.40.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{\csc^4(x)}{a + b\csc(x)}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(csc(x)^4/(a+b*csc(x)),x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.40.8 Giac [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 141, normalized size of antiderivative = 1.68

$$\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx = -\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)a^3}{\sqrt{-a^2+b^2}b^3} + \frac{b\tan\left(\frac{1}{2}x\right)^2-4\,a\tan\left(\frac{1}{2}x\right)}{8\,b^2} + \frac{(2\,a^2+b^2)\log\left(\left|\tan\left(\frac{1}{2}x\right)\right|\right)}{2\,b^3} - \frac{12\,a^2\tan\left(\frac{1}{2}x\right)^2+6\,b^2\tan\left(\frac{1}{2}x\right)^2-4\,ab\tan\left(\frac{1}{2}x\right)+b^2}{8\,b^3\tan\left(\frac{1}{2}x\right)^2}$$

input
```
integrate(csc(x)^4/(a+b*csc(x)),x, algorithm="giac")
```

output
```
-2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2
+ b^2)))*a^3/(sqrt(-a^2 + b^2)*b^3) + 1/8*(b*tan(1/2*x)^2 - 4*a*tan(1/2*x)
)/b^2 + 1/2*(2*a^2 + b^2)*log(abs(tan(1/2*x)))/b^3 - 1/8*(12*a^2*tan(1/2*x
)^2 + 6*b^2*tan(1/2*x)^2 - 4*a*b*tan(1/2*x) + b^2)/(b^3*tan(1/2*x)^2)
```

3.40.9 Mupad [B] (verification not implemented)

Time = 17.92 (sec) , antiderivative size = 515, normalized size of antiderivative = 6.13

$$\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx = -\frac{b^2\left(\frac{\cos(x)\sqrt{a^2-b^2}}{2}-\frac{\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)\sqrt{a^2-b^2}}{4}+\frac{\cos(2\,x)\,\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)\sqrt{a^2-b^2}}{4}\right)-\frac{a^2\,\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)\sqrt{a^2-b^2}}{2}-\frac{a\,b\,\sin(2\,x)\sqrt{a^2-b^2}}{2}}{}$$

input
```
int(1/(sin(x)^4*(a + b/sin(x))),x)
```

output
```
-(a^3*atan((a^4*sin(x/2)*(a^2 - b^2)^(1/2)*8i - b^4*sin(x/2)*(a^2 - b^2)^(
1/2)*1i + a*b^3*cos(x/2)*(a^2 - b^2)^(1/2)*1i + a^3*b*cos(x/2)*(a^2 - b^2)
^(1/2)*4i)/(b^5*cos(x/2) - 8*a^5*sin(x/2) + a^2*b^3*cos(x/2) + 4*a^3*b^2*s
in(x/2) - 4*a^4*b*cos(x/2) + 2*a*b^4*sin(x/2)))*1i + b^2*((cos(x)*(a^2 - b
^2)^(1/2))/2 - (log(sin(x/2)/cos(x/2))*(a^2 - b^2)^(1/2))/4 + (cos(2*x)*lo
g(sin(x/2)/cos(x/2))*(a^2 - b^2)^(1/2))/4) - (a^2*log(sin(x/2)/cos(x/2))*(
a^2 - b^2)^(1/2))/2 - a^3*cos(2*x)*atan((a^4*sin(x/2)*(a^2 - b^2)^(1/2)*8i
 - b^4*sin(x/2)*(a^2 - b^2)^(1/2)*1i + a*b^3*cos(x/2)*(a^2 - b^2)^(1/2)*1i
 + a^3*b*cos(x/2)*(a^2 - b^2)^(1/2)*4i)/(b^5*cos(x/2) - 8*a^5*sin(x/2) + a
^2*b^3*cos(x/2) + 4*a^3*b^2*sin(x/2) - 4*a^4*b*cos(x/2) + 2*a*b^4*sin(x/2)
))*1i - (a*b*sin(2*x)*(a^2 - b^2)^(1/2))/2 + (a^2*cos(2*x)*log(sin(x/2)/co
s(x/2))*(a^2 - b^2)^(1/2))/2)/((b^3*(a^2 - b^2)^(1/2))/2 - (b^3*cos(2*x)*(
a^2 - b^2)^(1/2))/2)
```

3.40. $\int \frac{\csc^4(x)}{a+b\csc(x)}\,dx$

3.41 $\int \frac{\csc^3(x)}{a+b\csc(x)}\,dx$

3.41.1 Optimal result

Integrand size = 13, antiderivative size = 62

$$\int \frac{\csc^3(x)}{a+b\csc(x)}\,dx=\frac{a\text{arctanh}(\cos(x))}{b^2}-\frac{2a^2\text{arctanh}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b^2\sqrt{a^2-b^2}}-\frac{\cot(x)}{b}$$

output
```
a*arctanh(cos(x))/b^2-cot(x)/b-2*a^2*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1
/2))/b^2/(a^2-b^2)^(1/2)
```

3.41.2 Mathematica [A] (verified)

Time = 0.37 (sec) , antiderivative size = 106, normalized size of antiderivative = 1.71

$$\int \frac{\csc^3(x)}{a+b\csc(x)}\,dx = \frac{\csc\left(\frac{x}{2}\right)\sec\left(\frac{x}{2}\right)\left(2a^2\arctan\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)\sin(x)+\sqrt{-a^2+b^2}\left(-b\cos(x)+a\left(\log\left(\cos\left(\frac{x}{2}\right)\right)-\log\left(\sin\left(\frac{x}{2}\right)\right)\right)\right)\right)}{2b^2\sqrt{-a^2+b^2}}$$

input
```
Integrate[Csc[x]^3/(a + b*Csc[x]),x]
```

output
```
(Csc[x/2]*Sec[x/2]*(2*a^2*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]]*Sin[x]
 + Sqrt[-a^2 + b^2]*(-(b*Cos[x]) + a*(Log[Cos[x/2]] - Log[Sin[x/2]])*Sin[x
])))/(2*b^2*Sqrt[-a^2 + b^2])
```

3.41.3 Rubi [A] (verified)

Time = 0.49 (sec) , antiderivative size = 76, normalized size of antiderivative = 1.23, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.846, Rules used = {3042, 4277, 3042, 4276, 3042, 4257, 4318, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc^3(x)}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\csc(x)^3}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{4277}$$

$$-\frac{a\int \frac{\csc^2(x)}{a+b\csc(x)}dx}{b}-\frac{\cot(x)}{b}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{a\int \frac{\csc(x)^2}{a+b\csc(x)}dx}{b}-\frac{\cot(x)}{b}$$

$$\Big\downarrow \text{4276}$$

$$-\frac{a\left(\frac{\int \csc(x)dx}{b}-\frac{a\int \frac{\csc(x)}{a+b\csc(x)}dx}{b}\right)}{b}-\frac{\cot(x)}{b}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{a\left(\frac{\int \csc(x)dx}{b}-\frac{a\int \frac{\csc(x)}{a+b\csc(x)}dx}{b}\right)}{b}-\frac{\cot(x)}{b}$$

$$\Big\downarrow \text{4257}$$

$$-\frac{a\left(-\frac{a\int \frac{\csc(x)}{a+b\csc(x)}dx}{b}-\frac{\text{arctanh}(\cos(x))}{b}\right)}{b}-\frac{\cot(x)}{b}$$

$$\Big\downarrow \text{4318}$$

$$-\frac{a\left(-\frac{a\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{b^2}-\frac{\text{arctanh}(\cos(x))}{b}\right)}{b}-\frac{\cot(x)}{b}$$

$$\big\downarrow \text{3042}$$

$$-\frac{a\left(-\frac{a\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{b^2}-\frac{\text{arctanh}(\cos(x))}{b}\right)}{b}-\frac{\cot(x)}{b}$$

$$\big\downarrow \text{3139}$$

$$-\frac{a\left(-\frac{2a\int \frac{1}{\tan^2\left(\frac{x}{2}\right)+\frac{2a\tan\left(\frac{x}{2}\right)}{b}+1}d\tan\left(\frac{x}{2}\right)}{b^2}-\frac{\text{arctanh}(\cos(x))}{b}\right)}{b}-\frac{\cot(x)}{b}$$

$$\big\downarrow \text{1083}$$

$$-\frac{a\left(\frac{4a\int \frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{b^2}-\frac{\text{arctanh}(\cos(x))}{b}\right)}{b}-\frac{\cot(x)}{b}$$

$$\big\downarrow \text{219}$$

$$-\frac{a\left(\frac{2a\,\text{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{2\sqrt{a^2-b^2}}\right)}{b\sqrt{a^2-b^2}}-\frac{\text{arctanh}(\cos(x))}{b}\right)}{b}-\frac{\cot(x)}{b}$$

input
```
Int[Csc[x]^3/(a + b*Csc[x]),x]
```

output
```
-((a*(-(ArcTanh[Cos[x]]/b) + (2*a*ArcTanh[(b*((2*a)/b + 2*Tan[x/2]))/(2*Sq
rt[a^2 - b^2])])/(b*Sqrt[a^2 - b^2])))/b) - Cot[x]/b
```

3.41.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4257
```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

rule 4276
```
Int[csc[(e_.) + (f_.)*(x_)]^2/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> Simp[1/b   Int[Csc[e + f*x], x], x] - Simp[a/b   Int[Csc[e + f*x]/(
a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

rule 4277
```
Int[csc[(e_.) + (f_.)*(x_)]^3/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> Simp[-Cot[e + f*x]/(b*f), x] - Simp[a/b   Int[Csc[e + f*x]^2/(a + b
*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

rule 4318
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[1/b   Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

3.41.4 Maple [A] (verified)

Time = 0.47 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.24

method	result	s
default	$\frac{\tan\left(\frac{x}{2}\right)}{2b}+\frac{2a^{2}\arctan\left(\frac{2b\tan\left(\frac{x}{2}\right)+2a}{2\sqrt{-a^{2}+b^{2}}}\right)}{b^{2}\sqrt{-a^{2}+b^{2}}}-\frac{1}{2b\tan\left(\frac{x}{2}\right)}-\frac{a\ln\left(\tan\left(\frac{x}{2}\right)\right)}{b^{2}}$	7
risch	$-\frac{2i}{b\left(\mathrm{e}^{2ix}-1\right)}+\frac{a\ln\left(\mathrm{e}^{ix}+1\right)}{b^{2}}-\frac{a\ln\left(\mathrm{e}^{ix}-1\right)}{b^{2}}+\frac{ia^{2}\ln\left(\mathrm{e}^{ix}+\frac{i\left(\sqrt{-a^{2}+b^{2}}\,b-a^{2}+b^{2}\right)}{a\sqrt{-a^{2}+b^{2}}}\right)}{\sqrt{-a^{2}+b^{2}}\,b^{2}}-\frac{ia^{2}\ln\left(\mathrm{e}^{ix}+\frac{i\left(\sqrt{-a^{2}+b^{2}}\,b+a^{2}-b^{2}\right)}{a\sqrt{-a^{2}+b^{2}}}\right)}{\sqrt{-a^{2}+b^{2}}\,b^{2}}$	1

input
```
int(csc(x)^3/(a+b*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
1/2*tan(1/2*x)/b+2*a^2/b^2/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*x)+2*a
)/(-a^2+b^2)^(1/2))-1/2/b/tan(1/2*x)-a/b^2*ln(tan(1/2*x))
```

3.41.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 131 vs. 2(56) = 112.

Time = 0.30 (sec) , antiderivative size = 308, normalized size of antiderivative = 4.97

$$\int \frac{\csc^3(x)}{a+b\csc(x)}\,dx = \left[\frac{\sqrt{a^2-b^2}a^2\log\left(-\frac{(a^2-2b^2)\cos(x)^2+2ab\sin(x)+a^2+b^2-2(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2ab\sin(x)-a^2-b^2}\right)\sin(x)+(a^3-ab^2)\log\left(\frac{1}{2}\,c\right.}{2(a^2b^2-b^4)\sin(x)}\right.$$
$$-\frac{2\sqrt{-a^2+b^2}a^2\arctan\left(-\frac{\sqrt{-a^2+b^2}(b\sin(x)+a)}{(a^2-b^2)\cos(x)}\right)\sin(x)-(a^3-ab^2)\log\left(\frac{1}{2}\cos(x)+\frac{1}{2}\right)\sin(x)+(a^3-ab^2)}{2(a^2b^2-b^4)\sin(x)}$$

input
```
integrate(csc(x)^3/(a+b*csc(x)),x, algorithm="fricas")
```

output
```
[1/2*(sqrt(a^2 - b^2)*a^2*log(-((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^
2 + b^2 - 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 -
2*a*b*sin(x) - a^2 - b^2))*sin(x) + (a^3 - a*b^2)*log(1/2*cos(x) + 1/2)*si
n(x) - (a^3 - a*b^2)*log(-1/2*cos(x) + 1/2)*sin(x) - 2*(a^2*b - b^3)*cos(x
))/((a^2*b^2 - b^4)*sin(x)), -1/2*(2*sqrt(-a^2 + b^2)*a^2*arctan(-sqrt(-a^
2 + b^2)*(b*sin(x) + a)/((a^2 - b^2)*cos(x)))*sin(x) - (a^3 - a*b^2)*log(1
/2*cos(x) + 1/2)*sin(x) + (a^3 - a*b^2)*log(-1/2*cos(x) + 1/2)*sin(x) + 2*
(a^2*b - b^3)*cos(x))/((a^2*b^2 - b^4)*sin(x))]
```

3.41.6 Sympy [F]

$$\int \frac{\csc^3(x)}{a+b\csc(x)}\,dx = \int \frac{\csc^3(x)}{a+b\csc(x)}\,dx$$

input
```
integrate(csc(x)**3/(a+b*csc(x)),x)
```

output
```
Integral(csc(x)**3/(a + b*csc(x)), x)
```

3.41.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{\csc^3(x)}{a + b\csc(x)}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(csc(x)^3/(a+b*csc(x)),x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.41.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 98, normalized size of antiderivative = 1.58

$$\int \frac{\csc^3(x)}{a + b\csc(x)}\,dx = \frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)a^2}{\sqrt{-a^2+b^2}b^2} - \frac{a\log\left(\left|\tan\left(\frac{1}{2}x\right)\right|\right)}{b^2} + \frac{\tan\left(\frac{1}{2}x\right)}{2b} + \frac{2a\tan\left(\frac{1}{2}x\right)-b}{2b^2\tan\left(\frac{1}{2}x\right)}$$

input
```
integrate(csc(x)^3/(a+b*csc(x)),x, algorithm="giac")
```

output
```
2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
 b^2)))*a^2/(sqrt(-a^2 + b^2)*b^2) - a*log(abs(tan(1/2*x)))/b^2 + 1/2*tan(
1/2*x)/b + 1/2*(2*a*tan(1/2*x) - b)/(b^2*tan(1/2*x))
```

3.41.9 Mupad [B] (verification not implemented)

Time = 18.11 (sec) , antiderivative size = 135, normalized size of antiderivative = 2.18

$$\int \frac{\csc^3(x)}{a + b\csc(x)}\,dx = -\frac{1}{b\tan(x)} - \frac{a\ln\left(\tan\left(\frac{x}{2}\right)\right)}{b^2} - \frac{a^2\,\mathrm{atan}\left(\frac{a^2\tan\left(\frac{x}{2}\right)\sqrt{a^2-b^2}\,4\mathrm{i}-b^2\tan\left(\frac{x}{2}\right)\sqrt{a^2-b^2}\,1\mathrm{i}+a\,b\sqrt{a^2-b^2}\,2\mathrm{i}}{4\tan\left(\frac{x}{2}\right)a^3+2\,a^2\,b-3\tan\left(\frac{x}{2}\right)a\,b^2-b^3}\right)\,2\mathrm{i}}{b^2\sqrt{a^2-b^2}}$$

input
```
int(1/(sin(x)^3*(a + b/sin(x))),x)
```

output
```
- 1/(b*tan(x)) - (a*log(tan(x/2)))/b^2 - (a^2*atan((a^2*tan(x/2)*(a^2 - b^
2)^(1/2)*4i - b^2*tan(x/2)*(a^2 - b^2)^(1/2)*1i + a*b*(a^2 - b^2)^(1/2)*2i
)/(4*a^3*tan(x/2) + 2*a^2*b - b^3 - 3*a*b^2*tan(x/2)))*2i)/(b^2*(a^2 - b^2
)^(1/2))
```

3.41. $\int \frac{\csc^3(x)}{a+b\csc(x)}\,dx$

3.42 $\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx$

3.42.1 Optimal result

Integrand size = 13, antiderivative size = 53

$$\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx = -\frac{\operatorname{arctanh}(\cos(x))}{b}+\frac{2a\operatorname{arctanh}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{b\sqrt{a^2-b^2}}$$

output
```
-arctanh(cos(x))/b+2*a*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/b/(a^2-b^
2)^(1/2)
```

3.42.2 Mathematica [A] (verified)

Time = 0.14 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.17

$$\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx = \frac{-\frac{2a\arctan\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}}-\log\left(\cos\left(\frac{x}{2}\right)\right)+\log\left(\sin\left(\frac{x}{2}\right)\right)}{b}$$

input
```
Integrate[Csc[x]^2/(a + b*Csc[x]),x]
```

output
```
((-2*a*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqrt[-a^2 + b^2] - Log[C
os[x/2]] + Log[Sin[x/2]])/b
```

3.42.3 Rubi [A] (verified)

Time = 0.40 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.17, number of steps used = 10, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.692, Rules used = {3042, 4276, 3042, 4257, 4318, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\csc(x)^2}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{4276}$$

$$\frac{\int \csc(x)dx}{b}-\frac{a\int \frac{\csc(x)}{a+b\csc(x)}dx}{b}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \csc(x)dx}{b}-\frac{a\int \frac{\csc(x)}{a+b\csc(x)}dx}{b}$$

$$\Big\downarrow \text{4257}$$

$$-\frac{a\int \frac{\csc(x)}{a+b\csc(x)}dx}{b}-\frac{\text{arctanh}(\cos(x))}{b}$$

$$\Big\downarrow \text{4318}$$

$$-\frac{a\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{b^2}-\frac{\text{arctanh}(\cos(x))}{b}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{a\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{b^2}-\frac{\text{arctanh}(\cos(x))}{b}$$

$$\Big\downarrow \text{3139}$$

$$-\frac{2a\int \frac{1}{\tan^2\left(\frac{x}{2}\right)+\frac{2a\tan\left(\frac{x}{2}\right)}{b}+1}d\tan\left(\frac{x}{2}\right)}{b^2}-\frac{\text{arctanh}(\cos(x))}{b}$$

$$\Big\downarrow \text{1083}$$

$$\frac{4a\int \frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{b^2}-\frac{\operatorname{arctanh}(\cos(x))}{b}$$

$$\Big\downarrow \ 219$$

$$\frac{2a\operatorname{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{2\sqrt{a^2-b^2}}\right)}{b\sqrt{a^2-b^2}}-\frac{\operatorname{arctanh}(\cos(x))}{b}$$

input
```
Int[Csc[x]^2/(a + b*Csc[x]),x]
```

output
```
-(ArcTanh[Cos[x]]/b) + (2*a*ArcTanh[(b*((2*a)/b + 2*Tan[x/2]))/(2*Sqrt[a^2
 - b^2])])/(b*Sqrt[a^2 - b^2])
```

3.42.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4257
```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

rule 4276
```
Int[csc[(e_.) + (f_.)*(x_)]^2/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Sym
bol] :> Simp[1/b   Int[Csc[e + f*x], x], x] - Simp[a/b   Int[Csc[e + f*x]/(
a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f}, x]
```

rule 4318
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[1/b   Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

3.42.4 Maple [A] (verified)

Time = 0.35 (sec) , antiderivative size = 53, normalized size of antiderivative = 1.00

method	result	size
default	$-\frac{2a \arctan\left(\frac{2b\tan\left(\frac{x}{2}\right)+2a}{2\sqrt{-a^2+b^2}}\right)}{b\sqrt{-a^2+b^2}}+\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{b}$	53
risch	$-\frac{a\ln\left(\mathrm{e}^{ix}+\frac{ib\sqrt{a^2-b^2}-a^2+b^2}{\sqrt{a^2-b^2}\,a}\right)}{\sqrt{a^2-b^2}\,b}+\frac{a\ln\left(\mathrm{e}^{ix}+\frac{ib\sqrt{a^2-b^2}+a^2-b^2}{\sqrt{a^2-b^2}\,a}\right)}{\sqrt{a^2-b^2}\,b}-\frac{\ln\left(\mathrm{e}^{ix}+1\right)}{b}+\frac{\ln\left(\mathrm{e}^{ix}-1\right)}{b}$	152

input
```
int(csc(x)^2/(a+b*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
-2*a/b/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*x)+2*a)/(-a^2+b^2)^(1/2))+
1/b*ln(tan(1/2*x))
```

3.42.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 100 vs. 2(47) = 94.

Time = 0.29 (sec) , antiderivative size = 245, normalized size of antiderivative = 4.62

$$\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx$$
$$= \left[\frac{\sqrt{a^2-b^2}a\log\left(\frac{(a^2-2b^2)\cos(x)^2+2ab\sin(x)+a^2+b^2+2(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2ab\sin(x)-a^2-b^2}\right)-(a^2-b^2)\log\left(\frac{1}{2}\cos(x)+\frac{1}{2}\right)}{2(a^2b-b^3)}\right.$$

input
```
integrate(csc(x)^2/(a+b*csc(x)),x, algorithm="fricas")
```

output
```
[1/2*(sqrt(a^2 - b^2)*a*log(((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2 +
 b^2 + 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2*a
*b*sin(x) - a^2 - b^2)) - (a^2 - b^2)*log(1/2*cos(x) + 1/2) + (a^2 - b^2)*
log(-1/2*cos(x) + 1/2))/(a^2*b - b^3), 1/2*(2*sqrt(-a^2 + b^2)*a*arctan(-s
qrt(-a^2 + b^2)*(b*sin(x) + a)/((a^2 - b^2)*cos(x))) - (a^2 - b^2)*log(1/2
*cos(x) + 1/2) + (a^2 - b^2)*log(-1/2*cos(x) + 1/2))/(a^2*b - b^3)]
```

3.42.6 Sympy [F]

$$\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx = \int \frac{\csc^2(x)}{a+b\csc(x)}\,dx$$

input
```
integrate(csc(x)**2/(a+b*csc(x)),x)
```

output
```
Integral(csc(x)**2/(a + b*csc(x)), x)
```

3.42.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(csc(x)^2/(a+b*csc(x)),x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.42.8 Giac [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 63, normalized size of antiderivative = 1.19

$$\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx = -\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)a}{\sqrt{-a^2+b^2}b}+\frac{\log\left(\left|\tan\left(\frac{1}{2}x\right)\right|\right)}{b}$$

input
```
integrate(csc(x)^2/(a+b*csc(x)),x, algorithm="giac")
```

output
```
-2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2
+ b^2)))*a/(sqrt(-a^2 + b^2)*b) + log(abs(tan(1/2*x)))/b
```

3.42.9 Mupad [B] (verification not implemented)

Time = 18.29 (sec) , antiderivative size = 129, normalized size of antiderivative = 2.43

$$\int \frac{\csc^2(x)}{a+b\csc(x)}\,dx = \frac{\ln\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)}{b}-\frac{2\,a\,\mathrm{atanh}\left(\frac{\sqrt{a^2-b^2}\,\left(4\mathrm{i}\,\sin\left(\frac{x}{2}\right)\,a^2+2\mathrm{i}\,\cos\left(\frac{x}{2}\right)\,a\,b-1\mathrm{i}\,\sin\left(\frac{x}{2}\right)\,b^2\right)}{a^3\,\sin\left(\frac{x}{2}\right)\,4\mathrm{i}+b\,\cos\left(\frac{x}{2}\right)\,\left(a^2-b^2\right)\,1\mathrm{i}+a^2\,b\,\cos\left(\frac{x}{2}\right)\,1\mathrm{i}-a\,b^2\,\sin\left(\frac{x}{2}\right)\,3\mathrm{i}}\right)}{b\,\sqrt{a^2-b^2}}$$

input
```
int(1/(sin(x)^2*(a + b/sin(x))),x)
```

output
```
log(sin(x/2)/cos(x/2))/b - (2*a*atanh(((a^2 - b^2)^(1/2)*(a^2*sin(x/2)*4i
- b^2*sin(x/2)*1i + a*b*cos(x/2)*2i))/(a^3*sin(x/2)*4i + b*cos(x/2)*(a^2 -
 b^2)*1i + a^2*b*cos(x/2)*1i - a*b^2*sin(x/2)*3i)))/(b*(a^2 - b^2)^(1/2))
```

3.43 $\int \frac{\csc(x)}{a+b\csc(x)}\,dx$

3.43.1 Optimal result

Integrand size = 11, antiderivative size = 40

$$\int \frac{\csc(x)}{a+b\csc(x)}\,dx=-\frac{2\text{arctanh}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{\sqrt{a^2-b^2}}$$

output
```
-2*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/(a^2-b^2)^(1/2)
```

3.43.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 40, normalized size of antiderivative = 1.00

$$\int \frac{\csc(x)}{a+b\csc(x)}\,dx=\frac{2\arctan\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}}$$

input
```
Integrate[Csc[x]/(a + b*Csc[x]),x]
```

output
```
(2*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqrt[-a^2 + b^2]
```

3.43.3 Rubi [A] (verified)

Time = 0.27 (sec) , antiderivative size = 49, normalized size of antiderivative = 1.22, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.545, Rules used = {3042, 4318, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(x)}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{\csc(x)}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{4318}$$

$$\frac{\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{b}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{b}$$

$$\Big\downarrow \text{3139}$$

$$\frac{2\int \frac{1}{\tan^2\left(\frac{x}{2}\right)+\frac{2a\tan\left(\frac{x}{2}\right)}{b}+1}d\tan\left(\frac{x}{2}\right)}{b}$$

$$\Big\downarrow \text{1083}$$

$$-\frac{4\int \frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{b}$$

$$\Big\downarrow \text{219}$$

$$-\frac{2\text{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{2\sqrt{a^2-b^2}}\right)}{\sqrt{a^2-b^2}}$$

input
```
Int[Csc[x]/(a + b*Csc[x]),x]
```

output
```
(-2*ArcTanh[(b*((2*a)/b + 2*Tan[x/2]))/(2*Sqrt[a^2 - b^2])])/Sqrt[a^2 - b^
2]
```

3.43.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4318
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[1/b   Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

3.43.4 Maple [A] (verified)

Time = 0.18 (sec) , antiderivative size = 39, normalized size of antiderivative = 0.98

method	result	size
default	$\frac{2\arctan\left(\frac{2b\tan\left(\frac{x}{2}\right)+2a}{2\sqrt{-a^{2}+b^{2}}}\right)}{\sqrt{-a^{2}+b^{2}}}$	39
risch	$-\frac{i\ln\left(\mathrm{e}^{ix}+\frac{i\left(\sqrt{-a^{2}+b^{2}}\,b+a^{2}-b^{2}\right)}{a\sqrt{-a^{2}+b^{2}}}\right)}{\sqrt{-a^{2}+b^{2}}}+\frac{i\ln\left(\mathrm{e}^{ix}+\frac{i\left(\sqrt{-a^{2}+b^{2}}\,b-a^{2}+b^{2}\right)}{a\sqrt{-a^{2}+b^{2}}}\right)}{\sqrt{-a^{2}+b^{2}}}$	122

input
```
int(csc(x)/(a+b*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
2/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*x)+2*a)/(-a^2+b^2)^(1/2))
```

3.43.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 154, normalized size of antiderivative = 3.85

$$\int \frac{\csc(x)}{a+b\csc(x)}\,dx = \left[\frac{\log\left(-\frac{(a^2-2\,b^2)\cos(x)^2+2\,ab\sin(x)+a^2+b^2-2\,(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2\,ab\sin(x)-a^2-b^2}\right)}{2\,\sqrt{a^2-b^2}},\; -\frac{\sqrt{-a^2+b^2}\arctan\left(-\frac{\sqrt{-a^2+b^2}(b\sin(x)+a)}{(a^2-b^2)\cos(x)}\right)}{a^2-b^2}\right]$$

input
```
integrate(csc(x)/(a+b*csc(x)),x, algorithm="fricas")
```

output
```
[1/2*log(-((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2 + b^2 - 2*(b*cos(x)
*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2*a*b*sin(x) - a^2 -
b^2))/sqrt(a^2 - b^2), -sqrt(-a^2 + b^2)*arctan(-sqrt(-a^2 + b^2)*(b*sin(x
) + a)/((a^2 - b^2)*cos(x)))/(a^2 - b^2)]
```

3.43.6 Sympy [F]

$$\int \frac{\csc(x)}{a+b\csc(x)}\,dx = \int \frac{\csc(x)}{a+b\csc(x)}\,dx$$

input
```
integrate(csc(x)/(a+b*csc(x)),x)
```

output
```
Integral(csc(x)/(a + b*csc(x)), x)
```

3.43.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{\csc(x)}{a+b\csc(x)}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(csc(x)/(a+b*csc(x)),x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.43.8 Giac [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 48, normalized size of antiderivative = 1.20

$$\int \frac{\csc(x)}{a+b\csc(x)}\,dx = \frac{2\left(\pi\left\lfloor \frac{x}{2\pi}+\frac{1}{2}\right\rfloor \operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)}{\sqrt{-a^2+b^2}}$$

input
```
integrate(csc(x)/(a+b*csc(x)),x, algorithm="giac")
```

output
```
2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
 b^2)))/sqrt(-a^2 + b^2)
```

3.43.9 Mupad [B] (verification not implemented)

Time = 18.13 (sec) , antiderivative size = 36, normalized size of antiderivative = 0.90

$$\int \frac{\csc(x)}{a+b\csc(x)}\,dx = -\frac{2\,\mathrm{atanh}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}\,\sqrt{a-b}}\right)}{\sqrt{a+b}\,\sqrt{a-b}}$$

input
```
int(1/(sin(x)*(a + b/sin(x))),x)
```

output
```
-(2*atanh((a + b*tan(x/2))/((a + b)^(1/2)*(a - b)^(1/2))))/((a + b)^(1/2)*
(a - b)^(1/2))
```

3.44 $\int \frac{1}{a+b\csc(c+dx)}\,dx$

3.44.1 Optimal result

Integrand size = 12, antiderivative size = 57

$$\int \frac{1}{a+b\csc(c+dx)}\,dx=\frac{x}{a}+\frac{2b\text{arctanh}\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{a^2-b^2}}\right)}{a\sqrt{a^2-b^2}d}$$

output
```
x/a+2*b*arctanh((a+b*tan(1/2*d*x+1/2*c))/(a^2-b^2)^(1/2))/a/d/(a^2-b^2)^(1
/2)
```

3.44.2 Mathematica [A] (verified)

Time = 0.21 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.04

$$\int \frac{1}{a+b\csc(c+dx)}\,dx=\frac{\frac{c}{d}+x-\frac{2b\arctan\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}d}}{a}$$

input
```
Integrate[(a + b*Csc[c + d*x])^(-1),x]
```

output
```
(c/d + x - (2*b*ArcTan[(a + b*Tan[(c + d*x)/2])/Sqrt[-a^2 + b^2]])/(Sqrt[-
a^2 + b^2]*d))/a
```

3.44.3 Rubi [A] (verified)

Time = 0.28 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.16, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.500, Rules used = {3042, 4270, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{a+b\csc(c+dx)}dx$$
$$\Big\downarrow \text{3042}$$
$$\int \frac{1}{a+b\csc(c+dx)}dx$$
$$\Big\downarrow \text{4270}$$
$$\frac{x}{a}-\frac{\int \frac{1}{\frac{a\sin(c+dx)}{b}+1}dx}{a}$$
$$\Big\downarrow \text{3042}$$
$$\frac{x}{a}-\frac{\int \frac{1}{\frac{a\sin(c+dx)}{b}+1}dx}{a}$$
$$\Big\downarrow \text{3139}$$
$$\frac{x}{a}-\frac{2\int \frac{1}{\tan^2\left(\frac{1}{2}(c+dx)\right)+\frac{2a\tan\left(\frac{1}{2}(c+dx)\right)}{b}+1}d\tan\left(\frac{1}{2}(c+dx)\right)}{ad}$$
$$\Big\downarrow \text{1083}$$
$$\frac{4\int \frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{1}{2}(c+dx)\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{1}{2}(c+dx)\right)\right)}{ad}+\frac{x}{a}$$
$$\Big\downarrow \text{219}$$
$$\frac{2b\operatorname{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{1}{2}(c+dx)\right)\right)}{2\sqrt{a^2-b^2}}\right)}{ad\sqrt{a^2-b^2}}+\frac{x}{a}$$

input
```
Int[(a + b*Csc[c + d*x])^(-1),x]
```

output
```
x/a + (2*b*ArcTanh[(b*((2*a)/b + 2*Tan[(c + d*x)/2]))/(2*Sqrt[a^2 - b^2])]
)/(a*Sqrt[a^2 - b^2]*d)
```

3.44.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4270
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(-1), x_Symbol] :> Simp[x/a, x]
- Simp[1/a   Int[1/(1 + (a/b)*Sin[c + d*x]), x], x] /; FreeQ[{a, b, c, d},
x] && NeQ[a^2 - b^2, 0]
```

3.44.4 Maple [A] (verified)

Time = 0.35 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.19

method	result	size
derivativedivides	$\frac{\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{a}-\frac{2b\arctan\left(\frac{2b\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+2a}{2\sqrt{-a^2+b^2}}\right)}{a\sqrt{-a^2+b^2}}}{d}$	68
default	$\frac{\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{a}-\frac{2b\arctan\left(\frac{2b\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+2a}{2\sqrt{-a^2+b^2}}\right)}{a\sqrt{-a^2+b^2}}}{d}$	68
risch	$\frac{x}{a}+\frac{b\ln\left(e^{i(dx+c)}+\frac{ib\sqrt{a^2-b^2}+a^2-b^2}{\sqrt{a^2-b^2}\,a}\right)}{\sqrt{a^2-b^2}\,da}-\frac{b\ln\left(e^{i(dx+c)}+\frac{ib\sqrt{a^2-b^2}-a^2+b^2}{\sqrt{a^2-b^2}\,a}\right)}{\sqrt{a^2-b^2}\,da}$	146

input
```
int(1/(a+b*csc(d*x+c)),x,method=_RETURNVERBOSE)
```

output
```
1/d*(2/a*arctan(tan(1/2*d*x+1/2*c))-2*b/a/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b
*tan(1/2*d*x+1/2*c)+2*a)/(-a^2+b^2)^(1/2)))
```

3.44.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 238, normalized size of antiderivative = 4.18

$$\int \frac{1}{a+b\csc(c+dx)}\,dx = \left[\frac{2\,(a^2-b^2)dx+\sqrt{a^2-b^2}b\log\left(\frac{(a^2-2\,b^2)\cos(dx+c)^2+2\,ab\sin(dx+c)+a^2+b^2+2\,(b\cos(dx+c)\sin(dx+c)+a\cos(dx+c))\sqrt{a^2-b^2}}{a^2\cos(dx+c)^2-2\,ab\sin(dx+c)-a^2-b^2}\right)}{2\,(a^3-ab^2)d}\right.$$

input
```
integrate(1/(a+b*csc(d*x+c)),x, algorithm="fricas")
```

output
```
[1/2*(2*(a^2 - b^2)*d*x + sqrt(a^2 - b^2)*b*log(((a^2 - 2*b^2)*cos(d*x + c
)^2 + 2*a*b*sin(d*x + c) + a^2 + b^2 + 2*(b*cos(d*x + c)*sin(d*x + c) + a*
cos(d*x + c))*sqrt(a^2 - b^2))/(a^2*cos(d*x + c)^2 - 2*a*b*sin(d*x + c) -
a^2 - b^2)))/((a^3 - a*b^2)*d), ((a^2 - b^2)*d*x + sqrt(-a^2 + b^2)*b*arct
an(-sqrt(-a^2 + b^2)*(b*sin(d*x + c) + a)/((a^2 - b^2)*cos(d*x + c))))/((a
^3 - a*b^2)*d)]
```

3.44.6 Sympy [F]

$$\int \frac{1}{a+b\csc(c+dx)}\,dx = \int \frac{1}{a+b\csc\left(c+dx\right)}\,dx$$

input
```
integrate(1/(a+b*csc(d*x+c)),x)
```

output
```
Integral(1/(a + b*csc(c + d*x)), x)
```

3.44.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{1}{a+b\csc(c+dx)}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(1/(a+b*csc(d*x+c)),x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.44.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.35

$$\int \frac{1}{a+b\csc(c+dx)}\,dx = -\frac{\frac{2\left(\pi\left\lfloor\frac{dx+c}{2\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+a}{\sqrt{-a^2+b^2}}\right)\right)b}{\sqrt{-a^2+b^2}\,a}-\frac{dx+c}{a}}{d}$$

input
```
integrate(1/(a+b*csc(d*x+c)),x, algorithm="giac")
```

output
```
-(2*(pi*floor(1/2*(d*x + c)/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*d*x + 1/2
*c) + a)/sqrt(-a^2 + b^2)))*b/(sqrt(-a^2 + b^2)*a) - (d*x + c)/a)/d
```

3.44.9 Mupad [B] (verification not implemented)

Time = 18.59 (sec) , antiderivative size = 184, normalized size of antiderivative = 3.23

$$\int \frac{1}{a+b\csc(c+dx)}\,dx = \frac{x}{a} - \frac{2\,b\,\mathrm{atanh}\left(\frac{2\,a^2\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,\left(a^2-b^2\right)-2\,b^4\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)-2\,b^2\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,\left(a^2-b^2\right)+a\,b^3\,\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)+3\,a^2\,b^2\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)+a\,b\,\cos\left(\frac{c}{2}+\right.}{a\,\left(2\,\sin\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a^2+b\,\cos\left(\frac{c}{2}+\frac{d\,x}{2}\right)\,a\right)\,\sqrt{a^2-b^2}}\right.}{a\,d\,\sqrt{a^2-b^2}}$$

input
```
int(1/(a + b/sin(c + d*x)),x)
```

output
```
x/a - (2*b*atanh((2*a^2*sin(c/2 + (d*x)/2)*(a^2 - b^2) - 2*b^4*sin(c/2 + (
d*x)/2) - 2*b^2*sin(c/2 + (d*x)/2)*(a^2 - b^2) + a*b^3*cos(c/2 + (d*x)/2)
+ 3*a^2*b^2*sin(c/2 + (d*x)/2) + a*b*cos(c/2 + (d*x)/2)*(a^2 - b^2))/(a*(2
*a^2*sin(c/2 + (d*x)/2) + a*b*cos(c/2 + (d*x)/2))*(a^2 - b^2)^(1/2))))/(a*
d*(a^2 - b^2)^(1/2))
```

3.44. $\int \frac{1}{a+b\csc(c+dx)}\,dx$

3.45 $\int \frac{\sin(x)}{a+b\csc(x)}\,dx$

3.45.1 Optimal result

Integrand size = 11, antiderivative size = 61

$$\int \frac{\sin(x)}{a+b\csc(x)}\,dx = -\frac{bx}{a^2} - \frac{2b^2\operatorname{arctanh}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^2\sqrt{a^2-b^2}} - \frac{\cos(x)}{a}$$

output
```
-b*x/a^2-cos(x)/a-2*b^2*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/a^2/(a^2
-b^2)^(1/2)
```

3.45.2 Mathematica [A] (verified)

Time = 0.27 (sec) , antiderivative size = 56, normalized size of antiderivative = 0.92

$$\int \frac{\sin(x)}{a+b\csc(x)}\,dx = -\frac{bx - \frac{2b^2\arctan\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}} + a\cos(x)}{a^2}$$

input
```
Integrate[Sin[x]/(a + b*Csc[x]),x]
```

output
```
-((b*x - (2*b^2*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqrt[-a^2 + b^2
] + a*Cos[x])/a^2)
```

3.45.3 Rubi [A] (verified)

Time = 0.36 (sec) , antiderivative size = 73, normalized size of antiderivative = 1.20, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}} = 0.909$, Rules used = {3042, 4340, 25, 27, 3042, 4270, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sin(x)}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\csc(x)(a+b\csc(x))}dx$$

$$\Big\downarrow \text{4340}$$

$$\frac{\int -\frac{b}{a+b\csc(x)}dx}{a}-\frac{\cos(x)}{a}$$

$$\Big\downarrow \text{25}$$

$$-\frac{\int \frac{b}{a+b\csc(x)}dx}{a}-\frac{\cos(x)}{a}$$

$$\Big\downarrow \text{27}$$

$$-\frac{b\int \frac{1}{a+b\csc(x)}dx}{a}-\frac{\cos(x)}{a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{b\int \frac{1}{a+b\csc(x)}dx}{a}-\frac{\cos(x)}{a}$$

$$\Big\downarrow \text{4270}$$

$$-\frac{b\left(\frac{x}{a}-\frac{\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{a}\right)}{a}-\frac{\cos(x)}{a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{b\left(\frac{x}{a}-\frac{\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{a}\right)}{a}-\frac{\cos(x)}{a}$$

$$\Big\downarrow \text{3139}$$

$$-\frac{b\left(\frac{x}{a}-\frac{2\int \frac{1}{\tan^2\left(\frac{x}{2}\right)+\frac{2a\tan\left(\frac{x}{2}\right)}{b}+1}d\tan\left(\frac{x}{2}\right)}{a}\right)}{a}-\frac{\cos(x)}{a}$$

$\downarrow$ 1083

$$-\frac{b\left(\frac{4\int \frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{a}+\frac{x}{a}\right)}{a}-\frac{\cos(x)}{a}$$

$\downarrow$ 219

$$-\frac{b\left(\frac{2b\,\text{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{2\sqrt{a^2-b^2}}\right)}{a\sqrt{a^2-b^2}}+\frac{x}{a}\right)}{a}-\frac{\cos(x)}{a}$$

input
```
Int[Sin[x]/(a + b*Csc[x]),x]
```

output
```
-((b*(x/a + (2*b*ArcTanh[(b*((2*a)/b + 2*Tan[x/2]))/(2*Sqrt[a^2 - b^2])])/
(a*Sqrt[a^2 - b^2])))/a) - Cos[x]/a
```

3.45.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

3.45. $\int \frac{\sin(x)}{a+b\csc(x)}\,dx$

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4270
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(-1), x_Symbol] :> Simp[x/a, x]
- Simp[1/a   Int[1/(1 + (a/b)*Sin[c + d*x]), x], x] /; FreeQ[{a, b, c, d},
x] && NeQ[a^2 - b^2, 0]
```

rule 4340
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[Cot[e + f*x]*((d*Csc[e + f*x])^n/(a*f*n)), x] - Sim
p[1/(a*d*n)   Int[((d*Csc[e + f*x])^(n + 1)/(a + b*Csc[e + f*x]))*Simp[b*n
- a*(n + 1)*Csc[e + f*x] - b*(n + 1)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a
, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1] && IntegerQ[2*n]
```

3.45.4 Maple [A] (verified)

Time = 0.46 (sec) , antiderivative size = 73, normalized size of antiderivative = 1.20

method	result	size
default	$\frac{2b^2\arctan\left(\frac{2b\tan\left(\frac{x}{2}\right)+2a}{2\sqrt{-a^2+b^2}}\right)}{a^2\sqrt{-a^2+b^2}}+\frac{-\frac{2a}{1+\tan\left(\frac{x}{2}\right)^2}-2b\arctan\left(\tan\left(\frac{x}{2}\right)\right)}{a^2}$	73
risch	$-\frac{xb}{a^2}-\frac{e^{ix}}{2a}-\frac{e^{-ix}}{2a}+\frac{ib^2\ln\left(e^{ix}+\frac{i\left(\sqrt{-a^2+b^2}\,b-a^2+b^2\right)}{a\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}\,a^2}-\frac{ib^2\ln\left(e^{ix}+\frac{i\left(\sqrt{-a^2+b^2}\,b+a^2-b^2\right)}{a\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}\,a^2}$	161

input
```
int(sin(x)/(a+b*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
2/a^2*b^2/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*x)+2*a)/(-a^2+b^2)^(1/2
))+2/a^2*(-a/(1+tan(1/2*x)^2)-b*arctan(tan(1/2*x)))
```

3.45.5 Fricas [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 235, normalized size of antiderivative = 3.85

$$\int \frac{\sin(x)}{a+b\csc(x)}\,dx$$
$$=\left[\frac{\sqrt{a^2-b^2}b^2\log\left(-\frac{(a^2-2\,b^2)\cos(x)^2+2\,ab\sin(x)+a^2+b^2-2\,(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2\,ab\sin(x)-a^2-b^2}\right)-2\,(a^2b-b^3)x-2\,(a^3-ab}{2\,(a^4-a^2b^2)}\right.$$
$$\left.-\frac{\sqrt{-a^2+b^2}b^2\arctan\left(-\frac{\sqrt{-a^2+b^2}(b\sin(x)+a)}{(a^2-b^2)\cos(x)}\right)+(a^2b-b^3)x+(a^3-ab^2)\cos(x)}{a^4-a^2b^2}\right]$$

input
```
integrate(sin(x)/(a+b*csc(x)),x, algorithm="fricas")
```

output
```
[1/2*(sqrt(a^2 - b^2)*b^2*log(-((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^
2 + b^2 - 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 -
2*a*b*sin(x) - a^2 - b^2)) - 2*(a^2*b - b^3)*x - 2*(a^3 - a*b^2)*cos(x))/(
a^4 - a^2*b^2), -(sqrt(-a^2 + b^2)*b^2*arctan(-sqrt(-a^2 + b^2)*(b*sin(x)
+ a)/((a^2 - b^2)*cos(x))) + (a^2*b - b^3)*x + (a^3 - a*b^2)*cos(x))/(a^4
- a^2*b^2)]
```

3.45.6 Sympy [F]

$$\int \frac{\sin(x)}{a+b\csc(x)}\,dx=\int \frac{\sin(x)}{a+b\csc(x)}\,dx$$

input
```
integrate(sin(x)/(a+b*csc(x)),x)
```

output
```
Integral(sin(x)/(a + b*csc(x)), x)
```

3.45.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{\sin(x)}{a+b\csc(x)}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(sin(x)/(a+b*csc(x)),x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.45.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.26

$$\int \frac{\sin(x)}{a+b\csc(x)}\,dx = \frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)b^2}{\sqrt{-a^2+b^2}\,a^2}-\frac{bx}{a^2}-\frac{2}{\left(\tan\left(\frac{1}{2}x\right)^2+1\right)a}$$

input
```
integrate(sin(x)/(a+b*csc(x)),x, algorithm="giac")
```

output
```
2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
 b^2)))*b^2/(sqrt(-a^2 + b^2)*a^2) - b*x/a^2 - 2/((tan(1/2*x)^2 + 1)*a)
```

3.45.9 Mupad [B] (verification not implemented)

Time = 19.00 (sec) , antiderivative size = 766, normalized size of antiderivative = 12.56

$$\int \frac{\sin(x)}{a+b\csc(x)}\,dx = -\frac{2}{a\left(\tan\left(\frac{x}{2}\right)^2+1\right)} - \frac{b\,x}{a^2}$$

$$-\frac{b^2\,\mathrm{atan}\left(\frac{\frac{b^2\sqrt{a^2-b^2}\left(\frac{32\,b^4}{a}-\frac{32\tan\left(\frac{x}{2}\right)\left(2\,a\,b^5-2\,a^3\,b^3\right)}{a^3}+\frac{b^2\sqrt{a^2-b^2}\left(32\,a^2\,b^2+64\,a\,b^3\tan\left(\frac{x}{2}\right)+\frac{b^2\sqrt{a^2-b^2}\left(32\,a^3\,b^2+\frac{32\tan\left(\frac{x}{2}\right)\left(3\,a^7\,b-2\cdots\right)}{a^3}\cdots\right)}{a^4-a^2\,b^2}\cdots\right)}{a^4-a^2\,b^2}\cdots\right)}{a^4-a^2\,b^2}\cdots}{\frac{128\,b^5\tan\left(\frac{x}{2}\right)}{a^3}+\frac{b^2\sqrt{a^2-b^2}\left(\frac{32\,b^4}{a}-\frac{32\tan\left(\frac{x}{2}\right)\left(2\,a\,b^5-2\,a^3\,b^3\right)}{a^3}+\frac{b^2\sqrt{a^2-b^2}\left(32\,a^2\,b^2+64\,a\,b^3\tan\left(\frac{x}{2}\right)+\frac{b^2\sqrt{a^2-b^2}\left(32\,a^3\,b^2+\frac{32\tan\left(\frac{x}{2}\right)\cdots}{\cdots}\cdots\right)}{a^4-a^2\,b^2}\cdots\right)}{a^4-a^2\,b^2}\cdots\right)}{a^4-a^2\,b^2}\cdots}\right)\cdots}{\cdots}$$

input
```
int(sin(x)/(a + b/sin(x)),x)
```

output
```
- 2/(a*(tan(x/2)^2 + 1)) - (b*x)/a^2 - (b^2*atan(((b^2*(a^2 - b^2)^(1/2)*(
(32*b^4)/a - (32*tan(x/2)*(2*a*b^5 - 2*a^3*b^3))/a^3 + (b^2*(a^2 - b^2)^(1
/2)*(32*a^2*b^2 + 64*a*b^3*tan(x/2) + (b^2*(a^2 - b^2)^(1/2)*(32*a^3*b^2 +
 (32*tan(x/2)*(3*a^7*b - 2*a^5*b^3))/a^3))/(a^4 - a^2*b^2)))/(a^4 - a^2*b^
2))*1i)/(a^4 - a^2*b^2) - (b^2*(a^2 - b^2)^(1/2)*((32*tan(x/2)*(2*a*b^5 -
2*a^3*b^3))/a^3 - (32*b^4)/a + (b^2*(a^2 - b^2)^(1/2)*(32*a^2*b^2 + 64*a*b
^3*tan(x/2) - (b^2*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (32*tan(x/2)*(3*a^7*b -
 2*a^5*b^3))/a^3))/(a^4 - a^2*b^2)))/(a^4 - a^2*b^2))*1i)/(a^4 - a^2*b^2))
/((128*b^5*tan(x/2))/a^3 + (b^2*(a^2 - b^2)^(1/2)*((32*b^4)/a - (32*tan(x/
2)*(2*a*b^5 - 2*a^3*b^3))/a^3 + (b^2*(a^2 - b^2)^(1/2)*(32*a^2*b^2 + 64*a*
b^3*tan(x/2) + (b^2*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (32*tan(x/2)*(3*a^7*b
- 2*a^5*b^3))/a^3))/(a^4 - a^2*b^2)))/(a^4 - a^2*b^2)))/(a^4 - a^2*b^2) +
(b^2*(a^2 - b^2)^(1/2)*((32*tan(x/2)*(2*a*b^5 - 2*a^3*b^3))/a^3 - (32*b^4)
/a + (b^2*(a^2 - b^2)^(1/2)*(32*a^2*b^2 + 64*a*b^3*tan(x/2) - (b^2*(a^2 -
b^2)^(1/2)*(32*a^3*b^2 + (32*tan(x/2)*(3*a^7*b - 2*a^5*b^3))/a^3))/(a^4 -
a^2*b^2)))/(a^4 - a^2*b^2)))/(a^4 - a^2*b^2)))*(a^2 - b^2)^(1/2)*2i)/(a^4
- a^2*b^2)
```

3.46 $\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx$

3.46.1 Optimal result

Integrand size = 13, antiderivative size = 82

$$\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx=\frac{(a^2+2b^2)\,x}{2a^3}+\frac{2b^3\text{arctanh}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^3\sqrt{a^2-b^2}}+\frac{b\cos(x)}{a^2}-\frac{\cos(x)\sin(x)}{2a}$$

output
```
1/2*(a^2+2*b^2)*x/a^3+b*cos(x)/a^2-1/2*cos(x)*sin(x)/a+2*b^3*arctanh((a+b*
tan(1/2*x))/(a^2-b^2)^(1/2))/a^3/(a^2-b^2)^(1/2)
```

3.46.2 Mathematica [A] (verified)

Time = 0.36 (sec) , antiderivative size = 78, normalized size of antiderivative = 0.95

$$\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx=\frac{2a^2x+4b^2x-\frac{8b^3\arctan\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}}+4ab\cos(x)-a^2\sin(2x)}{4a^3}$$

input
```
Integrate[Sin[x]^2/(a + b*Csc[x]),x]
```

output
```
(2*a^2*x + 4*b^2*x - (8*b^3*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])/Sqr
t[-a^2 + b^2] + 4*a*b*Cos[x] - a^2*Sin[2*x])/(4*a^3)
```

3.46.3 Rubi [A] (verified)

Time = 0.67 (sec) , antiderivative size = 103, normalized size of antiderivative = 1.26, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.000, Rules used = {3042, 4340, 25, 3042, 4592, 3042, 4407, 3042, 4318, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sin^2(x)}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\csc(x)^2(a+b\csc(x))}dx$$

$$\Big\downarrow \text{4340}$$

$$\frac{\int -\frac{(-b\csc^2(x)-a\csc(x)+2b)\sin(x)}{a+b\csc(x)}dx}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

$$\Big\downarrow \text{25}$$

$$-\frac{\int \frac{(-b\csc^2(x)-a\csc(x)+2b)\sin(x)}{a+b\csc(x)}dx}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{\int \frac{-b\csc(x)^2-a\csc(x)+2b}{\csc(x)(a+b\csc(x))}dx}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

$$\Big\downarrow \text{4592}$$

$$-\frac{-\frac{\int \frac{a^2+b\csc(x)a+2b^2}{a+b\csc(x)}dx}{a}-\frac{2b\cos(x)}{a}}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{-\frac{\int \frac{a^2+b\csc(x)a+2b^2}{a+b\csc(x)}dx}{a}-\frac{2b\cos(x)}{a}}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

$$\Big\downarrow \text{4407}$$

$$-\frac{-\frac{\frac{x(a^2+2b^2)}{a}-\frac{2b^3\int \frac{\csc(x)}{a+b\csc(x)}dx}{a}}{a}-\frac{2b\cos(x)}{a}}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{-\frac{\frac{x(a^2+2b^2)}{a}-\frac{2b^3\int\frac{\csc(x)}{a+b\csc(x)}dx}{a}}{a}-\frac{2b\cos(x)}{a}}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

$\Big\downarrow$ 4318

$$-\frac{-\frac{\frac{x(a^2+2b^2)}{a}-\frac{2b^2\int\frac{1}{\frac{a\sin(x)}{b}+1}dx}{a}}{a}-\frac{2b\cos(x)}{a}}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

$\Big\downarrow$ 3042

$$-\frac{-\frac{\frac{x(a^2+2b^2)}{a}-\frac{2b^2\int\frac{1}{\frac{a\sin(x)}{b}+1}dx}{a}}{a}-\frac{2b\cos(x)}{a}}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

$\Big\downarrow$ 3139

$$-\frac{-\frac{\frac{x(a^2+2b^2)}{a}-\frac{4b^2\int\frac{1}{\tan^2\left(\frac{x}{2}\right)+\frac{2a\tan\left(\frac{x}{2}\right)}{b}+1}d\tan\left(\frac{x}{2}\right)}{a}}{a}-\frac{2b\cos(x)}{a}}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

$\Big\downarrow$ 1083

$$-\frac{-\frac{\frac{8b^2\int\frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{a}+\frac{x(a^2+2b^2)}{a}}{a}-\frac{2b\cos(x)}{a}}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

$\Big\downarrow$ 219

$$-\frac{-\frac{\frac{4b^3\text{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{2\sqrt{a^2-b^2}}\right)}{a\sqrt{a^2-b^2}}+\frac{x(a^2+2b^2)}{a}}{a}-\frac{2b\cos(x)}{a}}{2a}-\frac{\sin(x)\cos(x)}{2a}$$

input
```
Int[Sin[x]^2/(a + b*Csc[x]),x]
```

output
```
-1/2*(-((((a^2 + 2*b^2)*x)/a + (4*b^3*ArcTanh[(b*((2*a)/b + 2*Tan[x/2]))/(
2*Sqrt[a^2 - b^2])])/(a*Sqrt[a^2 - b^2]))/a) - (2*b*Cos[x])/a)/a - (Cos[x]
*Sin[x])/(2*a)
```

3.46. $\int\frac{\sin^2(x)}{a+b\csc(x)}\,dx$

3.46.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4318
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[1/b   Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

rule 4340
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[Cot[e + f*x]*((d*Csc[e + f*x])^n/(a*f*n)), x] - Sim
p[1/(a*d*n)   Int[((d*Csc[e + f*x])^(n + 1)/(a + b*Csc[e + f*x]))*Simp[b*n
- a*(n + 1)*Csc[e + f*x] - b*(n + 1)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a
, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1] && IntegerQ[2*n]
```

rule 4407
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[c*(x/a), x] - Simp[(b*c - a*d)/a   Int[Csc[e + f*
x]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
- a*d, 0]
```

3.46. $\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx$

rule 4592
```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a
_))^(m_), x_Symbol] :> Simp[A*Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1)*((d
*Csc[e + f*x])^n/(a*f*n)), x] + Simp[1/(a*d*n)   Int[(a + b*Csc[e + f*x])^m
*(d*Csc[e + f*x])^(n + 1)*Simp[a*B*n - A*b*(m + n + 1) + a*(A + A*n + C*n)*
Csc[e + f*x] + A*b*(m + n + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a, b, d
, e, f, A, B, C, m}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1]
```

3.46.4 Maple [A] (verified)

Time = 0.54 (sec) , antiderivative size = 112, normalized size of antiderivative = 1.37

method	result	size
default	$\frac{\frac{2\left(\frac{a^{2}\tan\left(\frac{x}{2}\right)^{3}}{2}+ab\tan\left(\frac{x}{2}\right)^{2}-\frac{\tan\left(\frac{x}{2}\right)a^{2}}{2}+ab\right)}{\left(1+\tan\left(\frac{x}{2}\right)^{2}\right)^{2}}+\left(a^{2}+2b^{2}\right)\arctan\left(\tan\left(\frac{x}{2}\right)\right)}{a^{3}}-\frac{2b^{3}\arctan\left(\frac{2b\tan\left(\frac{x}{2}\right)+2a}{2\sqrt{-a^{2}+b^{2}}}\right)}{a^{3}\sqrt{-a^{2}+b^{2}}}$	112
risch	$\frac{x}{2a}+\frac{x\,b^{2}}{a^{3}}+\frac{b\,e^{ix}}{2a^{2}}+\frac{b\,e^{-ix}}{2a^{2}}-\frac{b^{3}\ln\left(e^{ix}+\frac{ib\sqrt{a^{2}-b^{2}}-a^{2}+b^{2}}{\sqrt{a^{2}-b^{2}}\,a}\right)}{\sqrt{a^{2}-b^{2}}\,a^{3}}+\frac{b^{3}\ln\left(e^{ix}+\frac{ib\sqrt{a^{2}-b^{2}}+a^{2}-b^{2}}{\sqrt{a^{2}-b^{2}}\,a}\right)}{\sqrt{a^{2}-b^{2}}\,a^{3}}-\frac{\sin(2x)}{4a}$	176

input
```
int(sin(x)^2/(a+b*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
2/a^3*((1/2*a^2*tan(1/2*x)^3+a*b*tan(1/2*x)^2-1/2*tan(1/2*x)*a^2+a*b)/(1+t
an(1/2*x)^2)^2+1/2*(a^2+2*b^2)*arctan(tan(1/2*x)))-2*b^3/a^3/(-a^2+b^2)^(1
/2)*arctan(1/2*(2*b*tan(1/2*x)+2*a)/(-a^2+b^2)^(1/2))
```

3.46.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 285, normalized size of antiderivative = 3.48

$$\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx$$
$$= \left[\frac{\sqrt{a^2-b^2}b^3\log\left(\frac{(a^2-2\,b^2)\cos(x)^2+2\,ab\sin(x)+a^2+b^2+2\,(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2\,ab\sin(x)-a^2-b^2}\right)-(a^4-a^2b^2)\cos(x)\sin(x)+}{2\,(a^5-a^3b^2)}\right.$$

input
```
integrate(sin(x)^2/(a+b*csc(x)),x, algorithm="fricas")
```

output
```
[1/2*(sqrt(a^2 - b^2)*b^3*log(((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) + a^2
 + b^2 + 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2 - 2
*a*b*sin(x) - a^2 - b^2)) - (a^4 - a^2*b^2)*cos(x)*sin(x) + (a^4 + a^2*b^2
 - 2*b^4)*x + 2*(a^3*b - a*b^3)*cos(x))/(a^5 - a^3*b^2), 1/2*(2*sqrt(-a^2
+ b^2)*b^3*arctan(-sqrt(-a^2 + b^2)*(b*sin(x) + a)/((a^2 - b^2)*cos(x))) -
 (a^4 - a^2*b^2)*cos(x)*sin(x) + (a^4 + a^2*b^2 - 2*b^4)*x + 2*(a^3*b - a*
b^3)*cos(x))/(a^5 - a^3*b^2)]
```

3.46.6 Sympy [F]

$$\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx = \int \frac{\sin^2{(x)}}{a+b\csc{(x)}}\,dx$$

input
```
integrate(sin(x)**2/(a+b*csc(x)),x)
```

output
```
Integral(sin(x)**2/(a + b*csc(x)), x)
```

3.46.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(sin(x)^2/(a+b*csc(x)),x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.46.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 112, normalized size of antiderivative = 1.37

$$\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx = -\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)b^3}{\sqrt{-a^2+b^2}a^3}+\frac{(a^2+2b^2)x}{2a^3}+\frac{a\tan\left(\frac{1}{2}x\right)^3+2b\tan\left(\frac{1}{2}x\right)^2-a\tan\left(\frac{1}{2}x\right)+2b}{\left(\tan\left(\frac{1}{2}x\right)^2+1\right)^2a^2}$$

input
```
integrate(sin(x)^2/(a+b*csc(x)),x, algorithm="giac")
```

output
```
-2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2
+ b^2)))*b^3/(sqrt(-a^2 + b^2)*a^3) + 1/2*(a^2 + 2*b^2)*x/a^3 + (a*tan(1/2
*x)^3 + 2*b*tan(1/2*x)^2 - a*tan(1/2*x) + 2*b)/((tan(1/2*x)^2 + 1)^2*a^2)
```

3.46.9 Mupad [B] (verification not implemented)

Time = 20.24 (sec) , antiderivative size = 1147, normalized size of antiderivative = 13.99

$$\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx = \text{Too large to display}$$

input
```
int(sin(x)^2/(a + b/sin(x)),x)
```

output
```
((2*b)/a^2 - tan(x/2)/a + tan(x/2)^3/a + (2*b*tan(x/2)^2)/a^2)/(2*tan(x/2)
^2 + tan(x/2)^4 + 1) - (atan((40*b^3*tan(x/2))/(8*a^2*b + 40*b^3 + (48*b^5
)/a^2) + (48*b^5*tan(x/2))/(8*a^4*b + 48*b^5 + 40*a^2*b^3) + (8*a*b*tan(x/
2))/(8*a*b + (40*b^3)/a + (48*b^5)/a^3))*(a^2*1i + b^2*2i)*1i)/a^3 + (b^3*
atan(((b^3*(a^2 - b^2)^(1/2)*((8*(4*a^2*b^6 + 4*a^4*b^4 + a^6*b^2))/a^5 +
(8*tan(x/2)*(2*a^8*b - 8*a^2*b^7 + 4*a^4*b^5 + 7*a^6*b^3))/a^6 + (b^3*(a^2
 - b^2)^(1/2)*(64*b^4*tan(x/2) + (8*(2*a^8*b + 2*a^6*b^3))/a^5 + (b^3*(a^2
 - b^2)^(1/2)*(32*a^3*b^2 + (8*tan(x/2)*(12*a^10*b - 8*a^8*b^3))/a^6))/(a^
5 - a^3*b^2)))/(a^5 - a^3*b^2))*1i)/(a^5 - a^3*b^2) + (b^3*(a^2 - b^2)^(1/
2)*((8*(4*a^2*b^6 + 4*a^4*b^4 + a^6*b^2))/a^5 + (8*tan(x/2)*(2*a^8*b - 8*a
^2*b^7 + 4*a^4*b^5 + 7*a^6*b^3))/a^6 - (b^3*(a^2 - b^2)^(1/2)*(64*b^4*tan(
x/2) + (8*(2*a^8*b + 2*a^6*b^3))/a^5 - (b^3*(a^2 - b^2)^(1/2)*(32*a^3*b^2
+ (8*tan(x/2)*(12*a^10*b - 8*a^8*b^3))/a^6))/(a^5 - a^3*b^2)))/(a^5 - a^3*
b^2))*1i)/(a^5 - a^3*b^2))/((16*(2*b^7 + a^2*b^5))/a^5 + (16*tan(x/2)*(8*b
^8 + 8*a^2*b^6 + 2*a^4*b^4))/a^6 + (b^3*(a^2 - b^2)^(1/2)*((8*(4*a^2*b^6 +
 4*a^4*b^4 + a^6*b^2))/a^5 + (8*tan(x/2)*(2*a^8*b - 8*a^2*b^7 + 4*a^4*b^5
+ 7*a^6*b^3))/a^6 + (b^3*(a^2 - b^2)^(1/2)*(64*b^4*tan(x/2) + (8*(2*a^8*b
+ 2*a^6*b^3))/a^5 + (b^3*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (8*tan(x/2)*(12*a
^10*b - 8*a^8*b^3))/a^6))/(a^5 - a^3*b^2)))/(a^5 - a^3*b^2)))/(a^5 - a^3*b
^2) - (b^3*(a^2 - b^2)^(1/2)*((8*(4*a^2*b^6 + 4*a^4*b^4 + a^6*b^2))/a^5...
```

3.46. $\int \frac{\sin^2(x)}{a+b\csc(x)}\,dx$

3.47 $\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx$

3.47.1 Optimal result

Integrand size = 13, antiderivative size = 110

$$\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx=-\frac{b(a^2+2b^2)\,x}{2a^4}-\frac{2b^4\operatorname{arctanh}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^4\sqrt{a^2-b^2}}-\frac{(2a^2+3b^2)\cos(x)}{3a^3}+\frac{b\cos(x)\sin(x)}{2a^2}-\frac{\cos(x)\sin^2(x)}{3a}$$

output
```
-1/2*b*(a^2+2*b^2)*x/a^4-1/3*(2*a^2+3*b^2)*cos(x)/a^3+1/2*b*cos(x)*sin(x)/
a^2-1/3*cos(x)*sin(x)^2/a-2*b^4*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/
a^4/(a^2-b^2)^(1/2)
```

3.47.2 Mathematica [A] (verified)

Time = 0.79 (sec) , antiderivative size = 98, normalized size of antiderivative = 0.89

$$\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx=\frac{-6b(a^2+2b^2)\,x+\frac{24b^4\arctan\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}}-3a(3a^2+4b^2)\cos(x)+a^3\cos(3x)+3a^2b\sin(2x)}{12a^4}$$

input
```
Integrate[Sin[x]^3/(a + b*Csc[x]),x]
```

output
```
(-6*b*(a^2 + 2*b^2)*x + (24*b^4*ArcTan[(a + b*Tan[x/2])/Sqrt[-a^2 + b^2]])
/Sqrt[-a^2 + b^2] - 3*a*(3*a^2 + 4*b^2)*Cos[x] + a^3*Cos[3*x] + 3*a^2*b*Si
n[2*x])/(12*a^4)
```

3.47.3 Rubi [A] (verified)

Time = 0.91 (sec) , antiderivative size = 136, normalized size of antiderivative = 1.24, number of steps used = 17, number of rules used = 16, $\frac{\text{number of rules}}{\text{integrand size}} = 1.231$, Rules used = {3042, 4340, 25, 3042, 4592, 3042, 4592, 27, 3042, 4407, 3042, 4318, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sin^3(x)}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\csc(x)^3(a+b\csc(x))}dx$$

$$\Big\downarrow \text{4340}$$

$$\frac{\int -\frac{\left(-2b\csc^2(x)-2a\csc(x)+3b\right)\sin^2(x)}{a+b\csc(x)}dx}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{25}$$

$$-\frac{\int \frac{\left(-2b\csc^2(x)-2a\csc(x)+3b\right)\sin^2(x)}{a+b\csc(x)}dx}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{\int \frac{-2b\csc(x)^2-2a\csc(x)+3b}{\csc(x)^2(a+b\csc(x))}dx}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{4592}$$

$$-\frac{-\frac{\int \frac{\left(-3b^2\csc^2(x)+ab\csc(x)+2\left(2a^2+3b^2\right)\right)\sin(x)}{a+b\csc(x)}dx}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{-\frac{\int \frac{-3b^2\csc(x)^2+ab\csc(x)+2\left(2a^2+3b^2\right)}{\csc(x)(a+b\csc(x))}dx}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{4592}$$

$$-\frac{-\frac{-\frac{\int \frac{3\left(a\csc(x)b^2+\left(a^2+2b^2\right)b\right)}{a+b\csc(x)}dx}{a}-\frac{2\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{27}$$

$$-\frac{-\frac{-\frac{3\int \frac{a\csc(x)b^2+\left(a^2+2b^2\right)b}{a+b\csc(x)}dx}{a}-\frac{2\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{-\frac{-\frac{3\int \frac{a\csc(x)b^2+\left(a^2+2b^2\right)b}{a+b\csc(x)}dx}{a}-\frac{2\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{4407}$$

$$-\frac{-\frac{-\frac{3\left(\frac{bx\left(a^2+2b^2\right)}{a}-\frac{2b^4\int \frac{\csc(x)}{a+b\csc(x)}dx}{a}\right)}{a}-\frac{2\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{-\frac{-\frac{3\left(\frac{bx\left(a^2+2b^2\right)}{a}-\frac{2b^4\int \frac{\csc(x)}{a+b\csc(x)}dx}{a}\right)}{a}-\frac{2\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{4318}$$

$$-\frac{-\frac{-\frac{3\left(\frac{bx\left(a^2+2b^2\right)}{a}-\frac{2b^3\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{a}\right)}{a}-\frac{2\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{-\frac{-\frac{3\left(\frac{bx\left(a^2+2b^2\right)}{a}-\frac{2b^3\int \frac{1}{\frac{a\sin(x)}{b}+1}dx}{a}\right)}{a}-\frac{2\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$$\Big\downarrow \text{3139}$$

3.47. $\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx$

$$-\frac{-\frac{-\frac{3\left(\frac{bx(a^2+2b^2)}{a}-\frac{4b^3\int \frac{1}{\tan^2\left(\frac{x}{2}\right)+\frac{2a\tan\left(\frac{x}{2}\right)}{b}+1}d\tan\left(\frac{x}{2}\right)}{a}\right)}{a}-\frac{2(2a^2+3b^2)\cos(x)}{a}}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$\Big\downarrow$ 1083

$$-\frac{-\frac{-\frac{3\left(\frac{8b^3\int \frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{a}+\frac{bx(a^2+2b^2)}{a}\right)}{a}-\frac{2(2a^2+3b^2)\cos(x)}{a}}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

$\Big\downarrow$ 219

$$-\frac{-\frac{-\frac{3\left(\frac{4b^4\text{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{2\sqrt{a^2-b^2}}\right)}{a\sqrt{a^2-b^2}}+\frac{bx(a^2+2b^2)}{a}\right)}{a}-\frac{2(2a^2+3b^2)\cos(x)}{a}}{2a}-\frac{3b\sin(x)\cos(x)}{2a}}{3a}-\frac{\sin^2(x)\cos(x)}{3a}$$

input
```
Int[Sin[x]^3/(a + b*Csc[x]),x]
```

output
```
-1/3*(Cos[x]*Sin[x]^2)/a - (-1/2*((-3*((b*(a^2 + 2*b^2)*x)/a + (4*b^4*ArcT
anh[(b*((2*a)/b + 2*Tan[x/2]))/(2*Sqrt[a^2 - b^2])])/(a*Sqrt[a^2 - b^2])))
/a - (2*(2*a^2 + 3*b^2)*Cos[x])/a)/a - (3*b*Cos[x]*Sin[x])/(2*a))/(3*a)
```

3.47.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4318
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[1/b   Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

rule 4340
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[Cot[e + f*x]*((d*Csc[e + f*x])^n/(a*f*n)), x] - Sim
p[1/(a*d*n)   Int[((d*Csc[e + f*x])^(n + 1)/(a + b*Csc[e + f*x]))*Simp[b*n
- a*(n + 1)*Csc[e + f*x] - b*(n + 1)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a
, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1] && IntegerQ[2*n]
```

rule 4407
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[c*(x/a), x] - Simp[(b*c - a*d)/a   Int[Csc[e + f*
x]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
- a*d, 0]
```

rule 4592
```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a
_))^(m_), x_Symbol] :> Simp[A*Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1)*((d
*Csc[e + f*x])^n/(a*f*n)), x] + Simp[1/(a*d*n)   Int[(a + b*Csc[e + f*x])^m
*(d*Csc[e + f*x])^(n + 1)*Simp[a*B*n - A*b*(m + n + 1) + a*(A + A*n + C*n)*
Csc[e + f*x] + A*b*(m + n + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a, b, d
, e, f, A, B, C, m}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1]
```

3.47. $\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx$

3.47.4 Maple [A] (verified)

Time = 0.72 (sec) , antiderivative size = 145, normalized size of antiderivative = 1.32

method	result
default	$\frac{2b^4 \arctan\left(\frac{2b\tan\left(\frac{x}{2}\right)+2a}{2\sqrt{-a^2+b^2}}\right)}{a^4\sqrt{-a^2+b^2}} + \frac{\frac{2\left(-\frac{a^2 b\tan\left(\frac{x}{2}\right)^5}{2}-a\,b^2\tan\left(\frac{x}{2}\right)^4+\left(-2a^3-2a\,b^2\right)\tan\left(\frac{x}{2}\right)^2+\frac{a^2 b\tan\left(\frac{x}{2}\right)}{2}-\frac{2a^3}{3}-a\,b^2\right)}{\left(1+\tan\left(\frac{x}{2}\right)^2\right)^3}-b(a^2+2b^2)\arctan}{a^4}$
risch	$-\frac{xb}{2a^2}-\frac{x\,b^3}{a^4}-\frac{3\,\mathrm{e}^{ix}}{8a}-\frac{\mathrm{e}^{ix}b^2}{2a^3}-\frac{3\,\mathrm{e}^{-ix}}{8a}-\frac{\mathrm{e}^{-ix}b^2}{2a^3}-\frac{ib^4\ln\left(\mathrm{e}^{ix}+\frac{i\left(\sqrt{-a^2+b^2}\,b+a^2-b^2\right)}{a\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}\,a^4}+\frac{ib^4\ln\left(\mathrm{e}^{ix}+\frac{i\left(\sqrt{-a^2+b^2}\,b-a^2+\right.}{a\sqrt{-a^2+b^2}}\right.}{\sqrt{-a^2+b^2}\,a^4}$

input
```
int(sin(x)^3/(a+b*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
2*b^4/a^4/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*x)+2*a)/(-a^2+b^2)^(1/2
))+2/a^4*((-1/2*a^2*b*tan(1/2*x)^5-a*b^2*tan(1/2*x)^4+(-2*a^3-2*a*b^2)*tan
(1/2*x)^2+1/2*a^2*b*tan(1/2*x)-2/3*a^3-a*b^2)/(1+tan(1/2*x)^2)^3-1/2*b*(a^
2+2*b^2)*arctan(tan(1/2*x)))
```

3.47.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 329, normalized size of antiderivative = 2.99

$$\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx$$
$$= \left[\frac{3\sqrt{a^2-b^2}b^4\log\left(-\frac{(a^2-2b^2)\cos(x)^2+2ab\sin(x)+a^2+b^2-2(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2ab\sin(x)-a^2-b^2}\right)+2\left(a^5-a^3b^2\right)\cos(x)^3+}{6\left(a^6-a^4b^2\right)}\right.$$
$$-\frac{6\sqrt{-a^2+b^2}b^4\arctan\left(-\frac{\sqrt{-a^2+b^2}(b\sin(x)+a)}{(a^2-b^2)\cos(x)}\right)-2\left(a^5-a^3b^2\right)\cos(x)^3-3\left(a^4b-a^2b^3\right)\cos(x)\sin(x)+3}{6\left(a^6-a^4b^2\right)}$$

input
```
integrate(sin(x)^3/(a+b*csc(x)),x, algorithm="fricas")
```

output
```
[1/6*(3*sqrt(a^2 - b^2)*b^4*log(-((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) +
a^2 + b^2 - 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2
- 2*a*b*sin(x) - a^2 - b^2)) + 2*(a^5 - a^3*b^2)*cos(x)^3 + 3*(a^4*b - a^2
*b^3)*cos(x)*sin(x) - 3*(a^4*b + a^2*b^3 - 2*b^5)*x - 6*(a^5 - a*b^4)*cos(
x))/(a^6 - a^4*b^2), -1/6*(6*sqrt(-a^2 + b^2)*b^4*arctan(-sqrt(-a^2 + b^2)
*(b*sin(x) + a)/((a^2 - b^2)*cos(x))) - 2*(a^5 - a^3*b^2)*cos(x)^3 - 3*(a^
4*b - a^2*b^3)*cos(x)*sin(x) + 3*(a^4*b + a^2*b^3 - 2*b^5)*x + 6*(a^5 - a*
b^4)*cos(x))/(a^6 - a^4*b^2)]
```

3.47.6 Sympy [F(-1)]

Timed out.

$$\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx = \text{Timed out}$$

input
```
integrate(sin(x)**3/(a+b*csc(x)),x)
```

output
```
Timed out
```

3.47.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(sin(x)^3/(a+b*csc(x)),x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.47.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 149, normalized size of antiderivative = 1.35

$$\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx = \frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)b^4}{\sqrt{-a^2+b^2}a^4} - \frac{(a^2b+2b^3)x}{2a^4} - \frac{3ab\tan\left(\frac{1}{2}x\right)^5+6b^2\tan\left(\frac{1}{2}x\right)^4+12a^2\tan\left(\frac{1}{2}x\right)^2+12b^2\tan\left(\frac{1}{2}x\right)^2-3ab\tan\left(\frac{1}{2}x\right)+4a^2+6b^2}{3\left(\tan\left(\frac{1}{2}x\right)^2+1\right)^3a^3}$$

input
```
integrate(sin(x)^3/(a+b*csc(x)),x, algorithm="giac")
```

output
```
2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2 +
 b^2)))*b^4/(sqrt(-a^2 + b^2)*a^4) - 1/2*(a^2*b + 2*b^3)*x/a^4 - 1/3*(3*a*
b*tan(1/2*x)^5 + 6*b^2*tan(1/2*x)^4 + 12*a^2*tan(1/2*x)^2 + 12*b^2*tan(1/2
*x)^2 - 3*a*b*tan(1/2*x) + 4*a^2 + 6*b^2)/((tan(1/2*x)^2 + 1)^3*a^3)
```

3.47.9 Mupad [B] (verification not implemented)

Time = 19.02 (sec) , antiderivative size = 1218, normalized size of antiderivative = 11.07

$$\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx = \text{Too large to display}$$

input
```
int(sin(x)^3/(a + b/sin(x)),x)
```

output
```
- ((2*(2*a^2 + 3*b^2))/(3*a^3) + (b*tan(x/2)^5)/a^2 + (2*b^2*tan(x/2)^4)/a
^3 + (4*tan(x/2)^2*(a^2 + b^2))/a^3 - (b*tan(x/2))/a^2)/(3*tan(x/2)^2 + 3*
tan(x/2)^4 + tan(x/2)^6 + 1) - (b^4*atan(((b^4*(a^2 - b^2)^(1/2)*((8*(4*a^
3*b^8 + 4*a^5*b^6 + a^7*b^4))/a^8 + (8*tan(x/2)*(4*a^5*b^7 - 8*a^3*b^9 + 7
*a^7*b^5 + 2*a^9*b^3))/a^9 + (b^4*(a^2 - b^2)^(1/2)*((8*(2*a^8*b^4 + 2*a^1
0*b^2))/a^8 + (64*b^5*tan(x/2))/a + (b^4*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (
8*tan(x/2)*(12*a^13*b - 8*a^11*b^3))/a^9))/(a^6 - a^4*b^2)))/(a^6 - a^4*b^
2))*1i)/(a^6 - a^4*b^2) + (b^4*(a^2 - b^2)^(1/2)*((8*(4*a^3*b^8 + 4*a^5*b^
6 + a^7*b^4))/a^8 + (8*tan(x/2)*(4*a^5*b^7 - 8*a^3*b^9 + 7*a^7*b^5 + 2*a^9
*b^3))/a^9 - (b^4*(a^2 - b^2)^(1/2)*((8*(2*a^8*b^4 + 2*a^10*b^2))/a^8 + (6
4*b^5*tan(x/2))/a - (b^4*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (8*tan(x/2)*(12*a
^13*b - 8*a^11*b^3))/a^9))/(a^6 - a^4*b^2)))/(a^6 - a^4*b^2))*1i)/(a^6 - a
^4*b^2))/((16*(2*b^10 + a^2*b^8))/a^8 + (16*tan(x/2)*(8*b^11 + 8*a^2*b^9 +
 2*a^4*b^7))/a^9 + (b^4*(a^2 - b^2)^(1/2)*((8*(4*a^3*b^8 + 4*a^5*b^6 + a^7
*b^4))/a^8 + (8*tan(x/2)*(4*a^5*b^7 - 8*a^3*b^9 + 7*a^7*b^5 + 2*a^9*b^3))/
a^9 + (b^4*(a^2 - b^2)^(1/2)*((8*(2*a^8*b^4 + 2*a^10*b^2))/a^8 + (64*b^5*t
an(x/2))/a + (b^4*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (8*tan(x/2)*(12*a^13*b -
 8*a^11*b^3))/a^9))/(a^6 - a^4*b^2)))/(a^6 - a^4*b^2)))/(a^6 - a^4*b^2) -
(b^4*(a^2 - b^2)^(1/2)*((8*(4*a^3*b^8 + 4*a^5*b^6 + a^7*b^4))/a^8 + (8*tan
(x/2)*(4*a^5*b^7 - 8*a^3*b^9 + 7*a^7*b^5 + 2*a^9*b^3))/a^9 - (b^4*(a^2 ...
```

3.47. $\int \frac{\sin^3(x)}{a+b\csc(x)}\,dx$

3.48 $\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx$

3.48.1 Optimal result

Integrand size = 13, antiderivative size = 144

$$\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx = \frac{(3a^4+4a^2b^2+8b^4)\,x}{8a^5} + \frac{2b^5\text{arctanh}\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right)}{a^5\sqrt{a^2-b^2}} + \frac{b(2a^2+3b^2)\cos(x)}{3a^4} - \frac{(3a^2+4b^2)\cos(x)\sin(x)}{8a^3} + \frac{b\cos(x)\sin^2(x)}{3a^2} - \frac{\cos(x)\sin^3(x)}{4a}$$

output
```
1/8*(3*a^4+4*a^2*b^2+8*b^4)*x/a^5+1/3*b*(2*a^2+3*b^2)*cos(x)/a^4-1/8*(3*a^
2+4*b^2)*cos(x)*sin(x)/a^3+1/3*b*cos(x)*sin(x)^2/a^2-1/4*cos(x)*sin(x)^3/a
+2*b^5*arctanh((a+b*tan(1/2*x))/(a^2-b^2)^(1/2))/a^5/(a^2-b^2)^(1/2)
```

3.48.2 Mathematica [A] (verified)

Time = 1.45 (sec) , antiderivative size = 129, normalized size of antiderivative = 0.90

$$\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx = \frac{36a^4x+48a^2b^2x+96b^4x-\frac{192b^5\arctan\left(\frac{a+b\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}}+24ab(3a^2+4b^2)\cos(x)-8a^3b\cos(3x)-24a^4\sin(2x)}{96a^5}$$

input
```
Integrate[Sin[x]^4/(a + b*Csc[x]),x]
```

output
```
(36*a^4*x + 48*a^2*b^2*x + 96*b^4*x - (192*b^5*ArcTan[(a + b*Tan[x/2])/Sqr
t[-a^2 + b^2]])/Sqrt[-a^2 + b^2] + 24*a*b*(3*a^2 + 4*b^2)*Cos[x] - 8*a^3*b
*Cos[3*x] - 24*a^4*Sin[2*x] - 24*a^2*b^2*Sin[2*x] + 3*a^4*Sin[4*x])/(96*a^
5)
```

3.48.3 Rubi [A] (verified)

Time = 1.27 (sec) , antiderivative size = 178, normalized size of antiderivative = 1.24, number of steps used = 19, number of rules used = 18, $\frac{\text{number of rules}}{\text{integrand size}} = 1.385$, Rules used = {3042, 4340, 25, 3042, 4592, 3042, 4592, 3042, 4592, 27, 3042, 4407, 3042, 4318, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sin^4(x)}{a+b\csc(x)}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{\csc(x)^4(a+b\csc(x))}dx$$

$$\Big\downarrow \text{4340}$$

$$\frac{\int -\frac{\left(-3b\csc^2(x)-3a\csc(x)+4b\right)\sin^3(x)}{a+b\csc(x)}dx}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$$\Big\downarrow \text{25}$$

$$-\frac{\int \frac{\left(-3b\csc^2(x)-3a\csc(x)+4b\right)\sin^3(x)}{a+b\csc(x)}dx}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{\int \frac{-3b\csc(x)^2-3a\csc(x)+4b}{\csc(x)^3(a+b\csc(x))}dx}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$$\Big\downarrow \text{4592}$$

$$-\frac{-\frac{\int \frac{\left(-8b^2\csc^2(x)+ab\csc(x)+3\left(3a^2+4b^2\right)\right)\sin^2(x)}{a+b\csc(x)}dx}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$$\Big\downarrow \text{3042}$$

$$-\frac{-\frac{\int \frac{-8b^2\csc(x)^2+ab\csc(x)+3\left(3a^2+4b^2\right)}{\csc(x)^2(a+b\csc(x))}dx}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$\downarrow$ 4592

$$-\frac{-\frac{-\frac{\int \frac{\left(-3b\left(3a^2+4b^2\right)\csc^2(x)-a\left(9a^2-4b^2\right)\csc(x)+8b\left(2a^2+3b^2\right)\right)\sin(x)}{a+b\csc(x)}dx}{2a}-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$\downarrow$ 3042

$$-\frac{-\frac{-\frac{\int \frac{-3b\left(3a^2+4b^2\right)\csc(x)^2-a\left(9a^2-4b^2\right)\csc(x)+8b\left(2a^2+3b^2\right)}{\csc(x)(a+b\csc(x))}dx}{2a}-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$\downarrow$ 4592

$$-\frac{-\frac{-\frac{-\frac{\int \frac{3\left(3a^4+4b^2a^2+b\left(3a^2+4b^2\right)\csc(x)a+8b^4\right)}{a+b\csc(x)}dx}{a}-\frac{8b\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$\downarrow$ 27

$$-\frac{-\frac{-\frac{-\frac{3\int \frac{3a^4+4b^2a^2+b\left(3a^2+4b^2\right)\csc(x)a+8b^4}{a+b\csc(x)}dx}{a}-\frac{8b\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$\downarrow$ 3042

$$-\frac{-\frac{-\frac{-\frac{3\int \frac{3a^4+4b^2a^2+b\left(3a^2+4b^2\right)\csc(x)a+8b^4}{a+b\csc(x)}dx}{a}-\frac{8b\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$\downarrow$ 4407

3.48. $\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx$

$$-\frac{-\frac{-\frac{-\frac{3\left(\frac{x\left(3a^4+4a^2b^2+8b^4\right)}{a}-\frac{8b^5\int\frac{\csc(x)}{a+b\csc(x)}dx}{a}\right)}{a}-\frac{8b\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$\Big\downarrow$ 3042

$$-\frac{-\frac{-\frac{-\frac{3\left(\frac{x\left(3a^4+4a^2b^2+8b^4\right)}{a}-\frac{8b^5\int\frac{\csc(x)}{a+b\csc(x)}dx}{a}\right)}{a}-\frac{8b\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$\Big\downarrow$ 4318

$$-\frac{-\frac{-\frac{-\frac{3\left(\frac{x\left(3a^4+4a^2b^2+8b^4\right)}{a}-\frac{8b^4\int\frac{1}{\frac{a\sin(x)}{b}+1}dx}{a}\right)}{a}-\frac{8b\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$\Big\downarrow$ 3042

$$-\frac{-\frac{-\frac{-\frac{3\left(\frac{x\left(3a^4+4a^2b^2+8b^4\right)}{a}-\frac{8b^4\int\frac{1}{\frac{a\sin(x)}{b}+1}dx}{a}\right)}{a}-\frac{8b\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$\Big\downarrow$ 3139

$$-\frac{-\frac{-\frac{-\frac{3\left(\frac{x\left(3a^4+4a^2b^2+8b^4\right)}{a}-\frac{16b^4\int\frac{1}{\tan^2\left(\frac{x}{2}\right)+\frac{2a\tan\left(\frac{x}{2}\right)}{b}+1}d\tan\left(\frac{x}{2}\right)}{a}\right)}{a}-\frac{8b\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-\frac{\sin^3(x)\cos(x)}{4a}$$

$\Big\downarrow$ 1083

3.48. $\int\frac{\sin^4(x)}{a+b\csc(x)}\,dx$

$$-\frac{-\frac{-\frac{-\frac{3\left(\frac{32b^4\int \frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{a}+\frac{x\left(3a^4+4a^2b^2+8b^4\right)}{a}\right)}{a}-\frac{8b\left(2a^2+3b^2\right)\cos(x)}{a}}{2a}-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}}{3a}-\frac{4b\sin^2(x)}{3a}}{4a}$$

$$\frac{\sin^3(x)\cos(x)}{4a}$$

$$\Big\downarrow \text{219}$$

$$-\frac{-\frac{-\frac{3\left(3a^2+4b^2\right)\sin(x)\cos(x)}{2a}-\frac{-\frac{8b\left(2a^2+3b^2\right)\cos(x)}{a}-\frac{3\left(\frac{16b^5\text{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{x}{2}\right)\right)}{2\sqrt{a^2-b^2}}\right)}{a\sqrt{a^2-b^2}}+\frac{x\left(3a^4+4a^2b^2+8b^4\right)}{a}\right)}{a}}{2a}}{3a}-\frac{4b\sin^2(x)\cos(x)}{3a}}{4a}-$$

$$\frac{\sin^3(x)\cos(x)}{4a}$$

input
```
Int[Sin[x]^4/(a + b*Csc[x]),x]
```

output
```
-1/4*(Cos[x]*Sin[x]^3)/a - ((-4*b*Cos[x]*Sin[x]^2)/(3*a) - (-1/2*((-3*(((3
*a^4 + 4*a^2*b^2 + 8*b^4)*x)/a + (16*b^5*ArcTanh[(b*((2*a)/b + 2*Tan[x/2])
)/(2*Sqrt[a^2 - b^2])])/(a*Sqrt[a^2 - b^2])))/a - (8*b*(2*a^2 + 3*b^2)*Cos
[x])/a)/a - (3*(3*a^2 + 4*b^2)*Cos[x]*Sin[x])/(2*a))/(3*a))/(4*a)
```

3.48.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

3.48. $\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx$

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4318
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[1/b   Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

rule 4340
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (
a_)), x_Symbol] :> Simp[Cot[e + f*x]*((d*Csc[e + f*x])^n/(a*f*n)), x] - Sim
p[1/(a*d*n)   Int[((d*Csc[e + f*x])^(n + 1)/(a + b*Csc[e + f*x]))*Simp[b*n
- a*(n + 1)*Csc[e + f*x] - b*(n + 1)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a
, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1] && IntegerQ[2*n]
```

rule 4407
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[c*(x/a), x] - Simp[(b*c - a*d)/a   Int[Csc[e + f*
x]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
- a*d, 0]
```

rule 4592
```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_)*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a
_))^(m_), x_Symbol] :> Simp[A*Cot[e + f*x]*(a + b*Csc[e + f*x])^(m + 1)*((d
*Csc[e + f*x])^n/(a*f*n)), x] + Simp[1/(a*d*n)   Int[(a + b*Csc[e + f*x])^m
*(d*Csc[e + f*x])^(n + 1)*Simp[a*B*n - A*b*(m + n + 1) + a*(A + A*n + C*n)*
Csc[e + f*x] + A*b*(m + n + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a, b, d
, e, f, A, B, C, m}, x] && NeQ[a^2 - b^2, 0] && LeQ[n, -1]
```

3.48. $\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx$

3.48.4 Maple [A] (verified)

Time = 0.98 (sec) , antiderivative size = 234, normalized size of antiderivative = 1.62

method	result
default	$-\frac{2b^5 \arctan\left(\frac{2b\tan\left(\frac{x}{2}\right)+2a}{2\sqrt{-a^2+b^2}}\right)}{a^5\sqrt{-a^2+b^2}}+\frac{2\left(\left(\frac{3}{8}a^4+\frac{1}{2}a^2b^2\right)\tan\left(\frac{x}{2}\right)^7+a\,b^3\tan\left(\frac{x}{2}\right)^6+\left(\frac{1}{2}a^2b^2+\frac{11}{8}a^4\right)\tan\left(\frac{x}{2}\right)^5+\left(2a^3b+3a\,b^3\right)\tan\left(\frac{x}{2}\right)^4+\left(-\frac{1}{2}a^2b^2\right.}{\left(1+\tan\left(\frac{x}{2}\right)^2\right)^4}$
risch	$\frac{3x}{8a}+\frac{x\,b^2}{2a^3}+\frac{x\,b^4}{a^5}+\frac{3b\,e^{ix}}{8a^2}+\frac{b^3e^{ix}}{2a^4}+\frac{3b\,e^{-ix}}{8a^2}+\frac{b^3e^{-ix}}{2a^4}+\frac{b^5\ln\left(e^{ix}+\frac{ib\sqrt{a^2-b^2}+a^2-b^2}{\sqrt{a^2-b^2}\,a}\right)}{\sqrt{a^2-b^2}\,a^5}-\frac{b^5\ln\left(e^{ix}+\frac{ib\sqrt{a^2-b^2}-a^2+b^2}{\sqrt{a^2-b^2}\,a}\right.}{\sqrt{a^2-b^2}\,a^5}$

input
```
int(sin(x)^4/(a+b*csc(x)),x,method=_RETURNVERBOSE)
```

output
```
-2*b^5/a^5/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*x)+2*a)/(-a^2+b^2)^(1/
2))+2/a^5*(((3/8*a^4+1/2*a^2*b^2)*tan(1/2*x)^7+a*b^3*tan(1/2*x)^6+(1/2*a^2
*b^2+11/8*a^4)*tan(1/2*x)^5+(2*a^3*b+3*a*b^3)*tan(1/2*x)^4+(-1/2*a^2*b^2-1
1/8*a^4)*tan(1/2*x)^3+(3*a*b^3+8/3*a^3*b)*tan(1/2*x)^2+(-3/8*a^4-1/2*a^2*b
^2)*tan(1/2*x)+2/3*a^3*b+a*b^3)/(1+tan(1/2*x)^2)^4+1/8*(3*a^4+4*a^2*b^2+8*
b^4)*arctan(tan(1/2*x)))
```

3.48.5 Fricas [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 410, normalized size of antiderivative = 2.85

$$\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx$$
$$=\left[\frac{12\sqrt{a^2-b^2}b^5\log\left(\frac{(a^2-2b^2)\cos(x)^2+2ab\sin(x)+a^2+b^2+2(b\cos(x)\sin(x)+a\cos(x))\sqrt{a^2-b^2}}{a^2\cos(x)^2-2ab\sin(x)-a^2-b^2}\right)-8\left(a^5b-a^3b^3\right)\cos(x)^3+\right.$$

input
```
integrate(sin(x)^4/(a+b*csc(x)),x, algorithm="fricas")
```

output
```
[1/24*(12*sqrt(a^2 - b^2)*b^5*log(((a^2 - 2*b^2)*cos(x)^2 + 2*a*b*sin(x) +
 a^2 + b^2 + 2*(b*cos(x)*sin(x) + a*cos(x))*sqrt(a^2 - b^2))/(a^2*cos(x)^2
 - 2*a*b*sin(x) - a^2 - b^2)) - 8*(a^5*b - a^3*b^3)*cos(x)^3 + 3*(3*a^6 +
a^4*b^2 + 4*a^2*b^4 - 8*b^6)*x + 24*(a^5*b - a*b^5)*cos(x) + 3*(2*(a^6 - a
^4*b^2)*cos(x)^3 - (5*a^6 - a^4*b^2 - 4*a^2*b^4)*cos(x))*sin(x))/(a^7 - a^
5*b^2), 1/24*(24*sqrt(-a^2 + b^2)*b^5*arctan(-sqrt(-a^2 + b^2)*(b*sin(x) +
 a)/((a^2 - b^2)*cos(x))) - 8*(a^5*b - a^3*b^3)*cos(x)^3 + 3*(3*a^6 + a^4*
b^2 + 4*a^2*b^4 - 8*b^6)*x + 24*(a^5*b - a*b^5)*cos(x) + 3*(2*(a^6 - a^4*b
^2)*cos(x)^3 - (5*a^6 - a^4*b^2 - 4*a^2*b^4)*cos(x))*sin(x))/(a^7 - a^5*b^
2)]
```

3.48.6 Sympy [F]

$$\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx=\int \frac{\sin^4\left(x\right)}{a+b\csc\left(x\right)}\,dx$$

input
```
integrate(sin(x)**4/(a+b*csc(x)),x)
```

output
```
Integral(sin(x)**4/(a + b*csc(x)), x)
```

3.48.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx=\text{Exception raised: ValueError}$$

input
```
integrate(sin(x)^4/(a+b*csc(x)),x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.48.8 Giac [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 252, normalized size of antiderivative = 1.75

$$\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx$$
$$= -\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}x\right)+a}{\sqrt{-a^2+b^2}}\right)\right)b^5}{\sqrt{-a^2+b^2}a^5}+\frac{(3a^4+4a^2b^2+8b^4)x}{8a^5}$$
$$+\frac{9a^3\tan\left(\frac{1}{2}x\right)^7+12ab^2\tan\left(\frac{1}{2}x\right)^7+24b^3\tan\left(\frac{1}{2}x\right)^6+33a^3\tan\left(\frac{1}{2}x\right)^5+12ab^2\tan\left(\frac{1}{2}x\right)^5+48a^2b\tan$$

input
```
integrate(sin(x)^4/(a+b*csc(x)),x, algorithm="giac")
```

output
```
-2*(pi*floor(1/2*x/pi + 1/2)*sgn(b) + arctan((b*tan(1/2*x) + a)/sqrt(-a^2
+ b^2)))*b^5/(sqrt(-a^2 + b^2)*a^5) + 1/8*(3*a^4 + 4*a^2*b^2 + 8*b^4)*x/a^
5 + 1/12*(9*a^3*tan(1/2*x)^7 + 12*a*b^2*tan(1/2*x)^7 + 24*b^3*tan(1/2*x)^6
 + 33*a^3*tan(1/2*x)^5 + 12*a*b^2*tan(1/2*x)^5 + 48*a^2*b*tan(1/2*x)^4 + 7
2*b^3*tan(1/2*x)^4 - 33*a^3*tan(1/2*x)^3 - 12*a*b^2*tan(1/2*x)^3 + 64*a^2*
b*tan(1/2*x)^2 + 72*b^3*tan(1/2*x)^2 - 9*a^3*tan(1/2*x) - 12*a*b^2*tan(1/2
*x) + 16*a^2*b + 24*b^3)/((tan(1/2*x)^2 + 1)^4*a^4)
```

3.48.9 Mupad [B] (verification not implemented)

Time = 19.32 (sec) , antiderivative size = 1639, normalized size of antiderivative = 11.38

$$\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx = \text{Too large to display}$$

input
```
int(sin(x)^4/(a + b/sin(x)),x)
```

output
```
((2*(2*a^2*b + 3*b^3))/(3*a^4) - (tan(x/2)*(3*a^2 + 4*b^2))/(4*a^3) + (tan
(x/2)^7*(3*a^2 + 4*b^2))/(4*a^3) - (tan(x/2)^3*(11*a^2 + 4*b^2))/(4*a^3) +
 (tan(x/2)^5*(11*a^2 + 4*b^2))/(4*a^3) + (2*b^3*tan(x/2)^6)/a^4 + (2*tan(x
/2)^4*(2*a^2*b + 3*b^3))/a^4 + (2*tan(x/2)^2*(8*a^2*b + 9*b^3))/(3*a^4))/(
4*tan(x/2)^2 + 6*tan(x/2)^4 + 4*tan(x/2)^6 + tan(x/2)^8 + 1) - (atan((81*b
^3*tan(x/2))/(8*((27*a^2*b)/8 + (81*b^3)/8 + (63*b^5)/(2*a^2) + (35*b^7)/a
^4 + (40*b^9)/a^6)) + (63*b^5*tan(x/2))/(2*((27*a^4*b)/8 + (63*b^5)/2 + (8
1*a^2*b^3)/8 + (35*b^7)/a^2 + (40*b^9)/a^4)) + (35*b^7*tan(x/2))/((27*a^6*
b)/8 + 35*b^7 + (63*a^2*b^5)/2 + (81*a^4*b^3)/8 + (40*b^9)/a^2) + (40*b^9*
tan(x/2))/((27*a^8*b)/8 + 40*b^9 + 35*a^2*b^7 + (63*a^4*b^5)/2 + (81*a^6*b
^3)/8) + (27*a*b*tan(x/2))/(8*((27*a*b)/8 + (81*b^3)/(8*a) + (63*b^5)/(2*a
^3) + (35*b^7)/a^5 + (40*b^9)/a^7)))*(a^4*3i + b^4*8i + a^2*b^2*4i)*1i)/(4
*a^5) + (b^5*atan(((b^5*(a^2 - b^2)^(1/2)*((32*a^4*b^10 + 32*a^6*b^8 + 32*
a^8*b^6 + 12*a^10*b^4 + (9*a^12*b^2)/2)/a^11 + (tan(x/2)*(18*a^14*b - 128*
a^4*b^11 + 64*a^6*b^9 + 64*a^8*b^7 + 104*a^10*b^5 + 39*a^12*b^3))/(2*a^12)
 + (b^5*(a^2 - b^2)^(1/2)*((12*a^14*b + 16*a^10*b^5 + 4*a^12*b^3)/a^11 + (
64*b^6*tan(x/2))/a^2 + (b^5*(a^2 - b^2)^(1/2)*(32*a^3*b^2 + (tan(x/2)*(192
*a^16*b - 128*a^14*b^3))/(2*a^12)))/(a^7 - a^5*b^2)))/(a^7 - a^5*b^2))*1i)
/(a^7 - a^5*b^2) + (b^5*(a^2 - b^2)^(1/2)*((32*a^4*b^10 + 32*a^6*b^8 + 32*
a^8*b^6 + 12*a^10*b^4 + (9*a^12*b^2)/2)/a^11 + (tan(x/2)*(18*a^14*b - 1...
```

3.48. $\int \frac{\sin^4(x)}{a+b\csc(x)}\,dx$

3.49 $\int \frac{1}{(a+b\csc(c+dx))^2}\,dx$

3.49.1 Optimal result

Integrand size = 12, antiderivative size = 108

$$\int \frac{1}{(a+b\csc(c+dx))^2}\,dx=\frac{x}{a^2}+\frac{2b(2a^2-b^2)\,\text{arctanh}\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{a^2-b^2}}\right)}{a^2\left(a^2-b^2\right)^{3/2}d}-\frac{b^2\cot(c+dx)}{a\left(a^2-b^2\right)d(a+b\csc(c+dx))}$$

output
```
x/a^2+2*b*(2*a^2-b^2)*arctanh((a+b*tan(1/2*d*x+1/2*c))/(a^2-b^2)^(1/2))/a^
2/(a^2-b^2)^(3/2)/d-b^2*cot(d*x+c)/a/(a^2-b^2)/d/(a+b*csc(d*x+c))
```

3.49.2 Mathematica [A] (verified)

Time = 0.65 (sec) , antiderivative size = 139, normalized size of antiderivative = 1.29

$$\int \frac{1}{(a+b\csc(c+dx))^2}\,dx=\frac{\csc(c+dx)\left(\frac{ab^2\cot(c+dx)}{(-a+b)(a+b)}+(c+dx)(a+b\csc(c+dx))-\frac{2b(-2a^2+b^2)\arctan\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{-a^2+b^2}}\right)(a+b\csc(c+dx))}{(-a^2+b^2)^{3/2}}\right)(b\ldots}{a^2d(a+b\csc(c+dx))^2}$$

input
```
Integrate[(a + b*Csc[c + d*x])^(-2),x]
```

output
```
(Csc[c + d*x]*((a*b^2*Cot[c + d*x])/((-a + b)*(a + b)) + (c + d*x)*(a + b*
Csc[c + d*x]) - (2*b*(-2*a^2 + b^2)*ArcTan[(a + b*Tan[(c + d*x)/2])/Sqrt[-
a^2 + b^2]]*(a + b*Csc[c + d*x]))/(-a^2 + b^2)^(3/2))*(b + a*Sin[c + d*x])
)/(a^2*d*(a + b*Csc[c + d*x])^2)
```

3.49.3 Rubi [A] (verified)

Time = 0.63 (sec) , antiderivative size = 142, normalized size of antiderivative = 1.31, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.917, Rules used = {3042, 4272, 25, 3042, 4407, 3042, 4318, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{(a+b\csc(c+dx))^2}dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{(a+b\csc(c+dx))^2}dx$$

$$\Big\downarrow \text{4272}$$

$$-\frac{\int -\frac{a^2-b\csc(c+dx)a-b^2}{a+b\csc(c+dx)}dx}{a\left(a^2-b^2\right)}-\frac{b^2\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int \frac{a^2-b\csc(c+dx)a-b^2}{a+b\csc(c+dx)}dx}{a\left(a^2-b^2\right)}-\frac{b^2\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{a^2-b\csc(c+dx)a-b^2}{a+b\csc(c+dx)}dx}{a\left(a^2-b^2\right)}-\frac{b^2\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}$$

$$\Big\downarrow \text{4407}$$

$$\frac{\frac{x\left(a^2-b^2\right)}{a}-\frac{b\left(2a^2-b^2\right)\int \frac{\csc(c+dx)}{a+b\csc(c+dx)}dx}{a}}{a\left(a^2-b^2\right)}-\frac{b^2\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\frac{x\left(a^2-b^2\right)}{a}-\frac{b\left(2a^2-b^2\right)\int \frac{\csc(c+dx)}{a+b\csc(c+dx)}dx}{a}}{a\left(a^2-b^2\right)}-\frac{b^2\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}$$

$$\Big\downarrow \text{4318}$$

$$\frac{\frac{x(a^2-b^2)}{a}-\frac{(2a^2-b^2)\int \frac{1}{\frac{a\sin(c+dx)}{b}+1}dx}{a}}{a\,(a^2-b^2)}-\frac{b^2\cot(c+dx)}{ad\,(a^2-b^2)\,(a+b\csc(c+dx))}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\frac{x(a^2-b^2)}{a}-\frac{(2a^2-b^2)\int \frac{1}{\frac{a\sin(c+dx)}{b}+1}dx}{a}}{a\,(a^2-b^2)}-\frac{b^2\cot(c+dx)}{ad\,(a^2-b^2)\,(a+b\csc(c+dx))}$$

$$\Big\downarrow \text{3139}$$

$$\frac{\frac{x(a^2-b^2)}{a}-\frac{2(2a^2-b^2)\int \frac{1}{\tan^2\left(\frac{1}{2}(c+dx)\right)+\frac{2a\tan\left(\frac{1}{2}(c+dx)\right)}{b}+1}d\tan\left(\frac{1}{2}(c+dx)\right)}{ad}}{a\,(a^2-b^2)}-\frac{b^2\cot(c+dx)}{ad\,(a^2-b^2)\,(a+b\csc(c+dx))}$$

$$\Big\downarrow \text{1083}$$

$$\frac{\frac{4(2a^2-b^2)\int \frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{1}{2}(c+dx)\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{1}{2}(c+dx)\right)\right)}{ad}+\frac{x(a^2-b^2)}{a}}{a\,(a^2-b^2)}-\frac{b^2\cot(c+dx)}{ad\,(a^2-b^2)\,(a+b\csc(c+dx))}$$

$$\Big\downarrow \text{219}$$

$$\frac{\frac{2b(2a^2-b^2)\text{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{1}{2}(c+dx)\right)\right)}{2\sqrt{a^2-b^2}}\right)}{ad\sqrt{a^2-b^2}}+\frac{x(a^2-b^2)}{a}}{a\,(a^2-b^2)}-\frac{b^2\cot(c+dx)}{ad\,(a^2-b^2)\,(a+b\csc(c+dx))}$$

input
```
Int[(a + b*Csc[c + d*x])^(-2),x]
```

output
```
(((a^2 - b^2)*x)/a + (2*b*(2*a^2 - b^2)*ArcTanh[(b*((2*a)/b + 2*Tan[(c + d
*x)/2]))/(2*Sqrt[a^2 - b^2])])/(a*Sqrt[a^2 - b^2]*d))/(a*(a^2 - b^2)) - (b
^2*Cot[c + d*x])/(a*(a^2 - b^2)*d*(a + b*Csc[c + d*x]))
```

3.49. $\int \frac{1}{(a+b\csc(c+dx))^2}\,dx$

3.49.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4272
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[b^2*Cot[
c + d*x]*((a + b*Csc[c + d*x])^(n + 1)/(a*d*(n + 1)*(a^2 - b^2))), x] + Sim
p[1/(a*(n + 1)*(a^2 - b^2))   Int[(a + b*Csc[c + d*x])^(n + 1)*Simp[(a^2 -
b^2)*(n + 1) - a*b*(n + 1)*Csc[c + d*x] + b^2*(n + 2)*Csc[c + d*x]^2, x], x
], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && LtQ[n, -1] && Integ
erQ[2*n]
```

rule 4318
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[1/b   Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

rule 4407
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[c*(x/a), x] - Simp[(b*c - a*d)/a   Int[Csc[e + f*
x]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
- a*d, 0]
```

3.49.4 Maple [A] (verified)

Time = 0.55 (sec) , antiderivative size = 168, normalized size of antiderivative = 1.56

method	result
derivativedivides	$\frac{-\frac{2b\left(\frac{\frac{a^2\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{2a^2-2b^2}+\frac{ab}{2a^2-2b^2}}{\frac{\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2 b}{2}+a\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+\frac{b}{2}}+\frac{2\left(2a^2-b^2\right)\arctan\left(\frac{2b\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+2a}{2\sqrt{-a^2+b^2}}\right)}{\left(2a^2-2b^2\right)\sqrt{-a^2+b^2}}\right)}{a^2}+\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{a^2}}{d}$
default	$\frac{-\frac{2b\left(\frac{\frac{a^2\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{2a^2-2b^2}+\frac{ab}{2a^2-2b^2}}{\frac{\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2 b}{2}+a\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+\frac{b}{2}}+\frac{2\left(2a^2-b^2\right)\arctan\left(\frac{2b\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+2a}{2\sqrt{-a^2+b^2}}\right)}{\left(2a^2-2b^2\right)\sqrt{-a^2+b^2}}\right)}{a^2}+\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{a^2}}{d}$
risch	$\frac{x}{a^2}-\frac{2ib^2\left(ia+b\,e^{i(dx+c)}\right)}{a^2(-a^2+b^2)d(2b\,e^{i(dx+c)}-ia\,e^{2i(dx+c)}+ia)}+\frac{2b\ln\left(e^{i(dx+c)}+\frac{ib\sqrt{a^2-b^2}+a^2-b^2}{\sqrt{a^2-b^2}\,a}\right)}{\sqrt{a^2-b^2}\,(a+b)(a-b)d}-\frac{b^3\ln\left(e^{i(dx+c)}+\frac{ib\sqrt{a^2}}{\sqrt{}}\right.}{\sqrt{a^2-b^2}\,(a+b)(a}$

input
```
int(1/(a+b*csc(d*x+c))^2,x,method=_RETURNVERBOSE)
```

output
```
1/d*(-2/a^2*b*((1/2*a^2/(a^2-b^2)*tan(1/2*d*x+1/2*c)+1/2*a*b/(a^2-b^2))/(1
/2*tan(1/2*d*x+1/2*c)^2*b+a*tan(1/2*d*x+1/2*c)+1/2*b)+2*(2*a^2-b^2)/(2*a^2
-2*b^2)/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*d*x+1/2*c)+2*a)/(-a^2+b^2
)^(1/2)))+2/a^2*arctan(tan(1/2*d*x+1/2*c)))
```

3.49.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 212 vs. 2(103) = 206.

Time = 0.28 (sec) , antiderivative size = 493, normalized size of antiderivative = 4.56

$$\int \frac{1}{(a+b\csc(c+dx))^2}\,dx$$
$$= \left[\frac{2\,(a^5-2\,a^3b^2+ab^4)dx\sin(dx+c)+2\,(a^4b-2\,a^2b^3+b^5)dx+(2\,a^2b^2-b^4+(2\,a^3b-ab^3)\sin(dx+c))}{2\,((a^7-2\,a^5b^2+a^3b^4)d\,s}\right.$$

input
```
integrate(1/(a+b*csc(d*x+c))^2,x, algorithm="fricas")
```

output
```
[1/2*(2*(a^5 - 2*a^3*b^2 + a*b^4)*d*x*sin(d*x + c) + 2*(a^4*b - 2*a^2*b^3
+ b^5)*d*x + (2*a^2*b^2 - b^4 + (2*a^3*b - a*b^3)*sin(d*x + c))*sqrt(a^2 -
 b^2)*log(((a^2 - 2*b^2)*cos(d*x + c)^2 + 2*a*b*sin(d*x + c) + a^2 + b^2 +
 2*(b*cos(d*x + c)*sin(d*x + c) + a*cos(d*x + c))*sqrt(a^2 - b^2))/(a^2*co
s(d*x + c)^2 - 2*a*b*sin(d*x + c) - a^2 - b^2)) - 2*(a^3*b^2 - a*b^4)*cos(
d*x + c))/((a^7 - 2*a^5*b^2 + a^3*b^4)*d*sin(d*x + c) + (a^6*b - 2*a^4*b^3
 + a^2*b^5)*d), ((a^5 - 2*a^3*b^2 + a*b^4)*d*x*sin(d*x + c) + (a^4*b - 2*a
^2*b^3 + b^5)*d*x + (2*a^2*b^2 - b^4 + (2*a^3*b - a*b^3)*sin(d*x + c))*sqr
t(-a^2 + b^2)*arctan(-sqrt(-a^2 + b^2)*(b*sin(d*x + c) + a)/((a^2 - b^2)*c
os(d*x + c))) - (a^3*b^2 - a*b^4)*cos(d*x + c))/((a^7 - 2*a^5*b^2 + a^3*b^
4)*d*sin(d*x + c) + (a^6*b - 2*a^4*b^3 + a^2*b^5)*d)]
```

3.49.6 Sympy [F]

$$\int \frac{1}{(a+b\csc(c+dx))^2}\,dx = \int \frac{1}{\left(a+b\csc\left(c+dx\right)\right)^2}\,dx$$

input
```
integrate(1/(a+b*csc(d*x+c))**2,x)
```

output
```
Integral((a + b*csc(c + d*x))**(-2), x)
```

3.49.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{1}{(a+b\csc(c+dx))^2}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(1/(a+b*csc(d*x+c))^2,x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.49.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 158, normalized size of antiderivative = 1.46

$$\int \frac{1}{(a+b\csc(c+dx))^2}\,dx = -\frac{\frac{2\,(2\,a^2b-b^3)\left(\pi\left\lfloor\frac{dx+c}{2\,\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+a}{\sqrt{-a^2+b^2}}\right)\right)}{(a^4-a^2b^2)\sqrt{-a^2+b^2}}+\frac{2\,(ab\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+b^2)}{(a^3-ab^2)\left(b\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^2+2\,a\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+b\right)}-\frac{dx+c}{a^2}}{d}$$

input
```
integrate(1/(a+b*csc(d*x+c))^2,x, algorithm="giac")
```

output
```
-(2*(2*a^2*b - b^3)*(pi*floor(1/2*(d*x + c)/pi + 1/2)*sgn(b) + arctan((b*t
an(1/2*d*x + 1/2*c) + a)/sqrt(-a^2 + b^2)))/((a^4 - a^2*b^2)*sqrt(-a^2 + b
^2)) + 2*(a*b*tan(1/2*d*x + 1/2*c) + b^2)/((a^3 - a*b^2)*(b*tan(1/2*d*x +
1/2*c)^2 + 2*a*tan(1/2*d*x + 1/2*c) + b)) - (d*x + c)/a^2)/d
```

3.49.9 Mupad [B] (verification not implemented)

Time = 22.69 (sec) , antiderivative size = 2677, normalized size of antiderivative = 24.79

$$\int \frac{1}{(a+b\csc(c+dx))^2}\,dx = \text{Too large to display}$$

input
```
int(1/(a + b/sin(c + d*x))^2,x)
```

output
```
(b*atan(((b*(2*a^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*((32*(a*b^6 - 2*a^3*
b^4 + a^5*b^2))/(a^6 + a^2*b^4 - 2*a^4*b^2) - (32*tan(c/2 + (d*x)/2)*(2*a*
b^7 - 2*a^7*b - 8*a^3*b^5 + 9*a^5*b^3))/(a^7 + a^3*b^4 - 2*a^5*b^2) + (b*(
2*a^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*((32*(a^8*b - a^6*b^3))/(a^6 + a^
2*b^4 - 2*a^4*b^2) + (32*tan(c/2 + (d*x)/2)*(2*a^4*b^6 - 6*a^6*b^4 + 4*a^8
*b^2))/(a^7 + a^3*b^4 - 2*a^5*b^2) + (b*(2*a^2 - b^2)*((a + b)^3*(a - b)^3
)^(1/2)*((32*(a^5*b^6 - 2*a^7*b^4 + a^9*b^2))/(a^6 + a^2*b^4 - 2*a^4*b^2)
+ (32*tan(c/2 + (d*x)/2)*(3*a^11*b - 2*a^5*b^7 + 7*a^7*b^5 - 8*a^9*b^3))/(
a^7 + a^3*b^4 - 2*a^5*b^2)))/(a^8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2)))/(a^
8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2))*1i)/(a^8 - a^2*b^6 + 3*a^4*b^4 - 3*a
^6*b^2) - (b*(2*a^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*((32*tan(c/2 + (d*x
)/2)*(2*a*b^7 - 2*a^7*b - 8*a^3*b^5 + 9*a^5*b^3))/(a^7 + a^3*b^4 - 2*a^5*b
^2) - (32*(a*b^6 - 2*a^3*b^4 + a^5*b^2))/(a^6 + a^2*b^4 - 2*a^4*b^2) + (b*
(2*a^2 - b^2)*((a + b)^3*(a - b)^3)^(1/2)*((32*(a^8*b - a^6*b^3))/(a^6 + a
^2*b^4 - 2*a^4*b^2) + (32*tan(c/2 + (d*x)/2)*(2*a^4*b^6 - 6*a^6*b^4 + 4*a^
8*b^2))/(a^7 + a^3*b^4 - 2*a^5*b^2) - (b*(2*a^2 - b^2)*((a + b)^3*(a - b)^
3)^(1/2)*((32*(a^5*b^6 - 2*a^7*b^4 + a^9*b^2))/(a^6 + a^2*b^4 - 2*a^4*b^2)
 + (32*tan(c/2 + (d*x)/2)*(3*a^11*b - 2*a^5*b^7 + 7*a^7*b^5 - 8*a^9*b^3))/
(a^7 + a^3*b^4 - 2*a^5*b^2)))/(a^8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2)))/(a
^8 - a^2*b^6 + 3*a^4*b^4 - 3*a^6*b^2))*1i)/(a^8 - a^2*b^6 + 3*a^4*b^4 -...
```

3.49. $\int \frac{1}{(a+b\csc(c+dx))^2}\,dx$

3.50 $\int \frac{1}{(a+b\csc(c+dx))^3}\,dx$

3.50.1 Optimal result

Integrand size = 12, antiderivative size = 170

$$\int \frac{1}{(a+b\csc(c+dx))^3}\,dx=\frac{x}{a^3}+\frac{b(6a^4-5a^2b^2+2b^4)\,\mathrm{arctanh}\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{a^2-b^2}}\right)}{a^3\left(a^2-b^2\right)^{5/2}d}-\frac{b^2\cot(c+dx)}{2a\left(a^2-b^2\right)d(a+b\csc(c+dx))^2}-\frac{b^2(5a^2-2b^2)\cot(c+dx)}{2a^2\left(a^2-b^2\right)^2d(a+b\csc(c+dx))}$$

output
```
x/a^3+b*(6*a^4-5*a^2*b^2+2*b^4)*arctanh((a+b*tan(1/2*d*x+1/2*c))/(a^2-b^2)
^(1/2))/a^3/(a^2-b^2)^(5/2)/d-1/2*b^2*cot(d*x+c)/a/(a^2-b^2)/d/(a+b*csc(d*
x+c))^2-1/2*b^2*(5*a^2-2*b^2)*cot(d*x+c)/a^2/(a^2-b^2)^2/d/(a+b*csc(d*x+c)
)
```

3.50.2 Mathematica [A] (verified)

Time = 1.97 (sec) , antiderivative size = 216, normalized size of antiderivative = 1.27

$$\int \frac{1}{(a+b\csc(c+dx))^3}\,dx$$

$$=\frac{\csc^2(c+dx)(b+a\sin(c+dx))\left(\frac{ab^3\cot(c+dx)}{(a-b)(a+b)}-\frac{3ab^2(2a^2-b^2)\cot(c+dx)(b+a\sin(c+dx))}{(a-b)^2(a+b)^2}+2(c+dx)\csc(c+dx)(b-\right.}{2a^3d(a+b\csc(c+dx))^3}$$

input
```
Integrate[(a + b*Csc[c + d*x])^(-3),x]
```

output
```
(Csc[c + d*x]^2*(b + a*Sin[c + d*x])*((a*b^3*Cot[c + d*x])/((a - b)*(a + b
)) - (3*a*b^2*(2*a^2 - b^2)*Cot[c + d*x]*(b + a*Sin[c + d*x]))/((a - b)^2*
(a + b)^2) + 2*(c + d*x)*Csc[c + d*x]*(b + a*Sin[c + d*x])^2 - (2*b*(6*a^4
 - 5*a^2*b^2 + 2*b^4)*ArcTan[(a + b*Tan[(c + d*x)/2])/Sqrt[-a^2 + b^2]]*Cs
c[c + d*x]*(b + a*Sin[c + d*x])^2)/(-a^2 + b^2)^(5/2)))/(2*a^3*d*(a + b*Cs
c[c + d*x])^3)
```

3.50.3 Rubi [A] (verified)

Time = 0.99 (sec) , antiderivative size = 225, normalized size of antiderivative = 1.32, number of steps used = 15, number of rules used = 14, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.167, Rules used = {3042, 4272, 25, 3042, 4548, 25, 3042, 4407, 3042, 4318, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{(a+b\csc(c+dx))^3}\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{1}{(a+b\csc(c+dx))^3}dx$$

$$\Big\downarrow \text{4272}$$

$$-\frac{\int -\frac{b^2\csc^2(c+dx)-2ab\csc(c+dx)+2(a^2-b^2)}{(a+b\csc(c+dx))^2}dx}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int \frac{b^2\csc^2(c+dx)-2ab\csc(c+dx)+2(a^2-b^2)}{(a+b\csc(c+dx))^2}dx}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \frac{b^2\csc(c+dx)^2-2ab\csc(c+dx)+2(a^2-b^2)}{(a+b\csc(c+dx))^2}dx}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$$\Big\downarrow \text{4548}$$

$$-\frac{\frac{\int -\frac{2(a^2-b^2)^2-ab(4a^2-b^2)\csc(c+dx)}{a+b\csc(c+dx)}dx}{a(a^2-b^2)}-\frac{b^2(5a^2-2b^2)\cot(c+dx)}{ad(a^2-b^2)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$$\Big\downarrow \; 25$$

$$\frac{\frac{\int \frac{2(a^2-b^2)^2-ab(4a^2-b^2)\csc(c+dx)}{a+b\csc(c+dx)}dx}{a(a^2-b^2)}-\frac{b^2(5a^2-2b^2)\cot(c+dx)}{ad(a^2-b^2)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$$\Big\downarrow \; 3042$$

$$\frac{\frac{\int \frac{2(a^2-b^2)^2-ab(4a^2-b^2)\csc(c+dx)}{a+b\csc(c+dx)}dx}{a(a^2-b^2)}-\frac{b^2(5a^2-2b^2)\cot(c+dx)}{ad(a^2-b^2)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$$\Big\downarrow \; 4407$$

$$\frac{\frac{\frac{2x(a^2-b^2)^2}{a}-\frac{b(6a^4-5a^2b^2+2b^4)\int \frac{\csc(c+dx)}{a+b\csc(c+dx)}dx}{a}}{a(a^2-b^2)}-\frac{b^2(5a^2-2b^2)\cot(c+dx)}{ad(a^2-b^2)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$$\Big\downarrow \; 3042$$

$$\frac{\frac{\frac{2x(a^2-b^2)^2}{a}-\frac{b(6a^4-5a^2b^2+2b^4)\int \frac{\csc(c+dx)}{a+b\csc(c+dx)}dx}{a}}{a(a^2-b^2)}-\frac{b^2(5a^2-2b^2)\cot(c+dx)}{ad(a^2-b^2)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$$\Big\downarrow \; 4318$$

$$\frac{\frac{\frac{2x(a^2-b^2)^2}{a}-\frac{(6a^4-5a^2b^2+2b^4)\int \frac{1}{\frac{a\sin(c+dx)}{b}+1}dx}{a}}{a(a^2-b^2)}-\frac{b^2(5a^2-2b^2)\cot(c+dx)}{ad(a^2-b^2)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$$\Big\downarrow \; 3042$$

$$\frac{\frac{\frac{2x(a^2-b^2)^2}{a}-\frac{(6a^4-5a^2b^2+2b^4)\int \frac{1}{\frac{a\sin(c+dx)}{b}+1}dx}{a}}{a(a^2-b^2)}-\frac{b^2(5a^2-2b^2)\cot(c+dx)}{ad(a^2-b^2)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$$\Big\downarrow \; 3139$$

3.50. $\int \frac{1}{(a+b\csc(c+dx))^3}\,dx$

$$\frac{\frac{\frac{2x(a^2-b^2)^2}{a}-\frac{2(6a^4-5a^2b^2+2b^4)\int \frac{1}{\tan^2(\frac{1}{2}(c+dx))+\frac{2a\tan(\frac{1}{2}(c+dx))}{b}+1}d\tan(\frac{1}{2}(c+dx))}{ad}}{a(a^2-b^2)}-\frac{b^2(5a^2-2b^2)\cot(c+dx)}{ad(a^2-b^2)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$\Big\downarrow$ 1083

$$\frac{\frac{\frac{4(6a^4-5a^2b^2+2b^4)\int \frac{1}{-(\frac{2a}{b}+2\tan(\frac{1}{2}(c+dx)))^2-4(1-\frac{a^2}{b^2})}d(\frac{2a}{b}+2\tan(\frac{1}{2}(c+dx)))}{ad}+\frac{2x(a^2-b^2)^2}{a}}{a(a^2-b^2)}-\frac{b^2(5a^2-2b^2)\cot(c+dx)}{ad(a^2-b^2)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

$\Big\downarrow$ 219

$$\frac{\frac{\frac{2x(a^2-b^2)^2}{a}+\frac{2b(6a^4-5a^2b^2+2b^4)\text{arctanh}\left(\frac{b(\frac{2a}{b}+2\tan(\frac{1}{2}(c+dx)))}{2\sqrt{a^2-b^2}}\right)}{ad\sqrt{a^2-b^2}}}{a(a^2-b^2)}-\frac{b^2(5a^2-2b^2)\cot(c+dx)}{ad(a^2-b^2)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}$$

input
```
Int[(a + b*Csc[c + d*x])^(-3),x]
```

output
```
-1/2*(b^2*Cot[c + d*x])/(a*(a^2 - b^2)*d*(a + b*Csc[c + d*x])^2) + (((2*(a
^2 - b^2)^2*x)/a + (2*b*(6*a^4 - 5*a^2*b^2 + 2*b^4)*ArcTanh[(b*((2*a)/b +
2*Tan[(c + d*x)/2]))/(2*Sqrt[a^2 - b^2])])/(a*Sqrt[a^2 - b^2]*d))/(a*(a^2
- b^2)) - (b^2*(5*a^2 - 2*b^2)*Cot[c + d*x])/(a*(a^2 - b^2)*d*(a + b*Csc[c
 + d*x])))/(2*a*(a^2 - b^2))
```

3.50.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4272
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[b^2*Cot[
c + d*x]*((a + b*Csc[c + d*x])^(n + 1)/(a*d*(n + 1)*(a^2 - b^2))), x] + Sim
p[1/(a*(n + 1)*(a^2 - b^2))   Int[(a + b*Csc[c + d*x])^(n + 1)*Simp[(a^2 -
b^2)*(n + 1) - a*b*(n + 1)*Csc[c + d*x] + b^2*(n + 2)*Csc[c + d*x]^2, x], x
], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && LtQ[n, -1] && Integ
erQ[2*n]
```

rule 4318
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[1/b   Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

rule 4407
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[c*(x/a), x] - Simp[(b*c - a*d)/a   Int[Csc[e + f*
x]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
- a*d, 0]
```

rule 4548
```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_Symbol] :> Simp[(A*b^2 -
a*b*B + a^2*C)*Cot[e + f*x]*((a + b*Csc[e + f*x])^(m + 1)/(a*f*(m + 1)*(a^2
 - b^2))), x] + Simp[1/(a*(m + 1)*(a^2 - b^2))   Int[(a + b*Csc[e + f*x])^(
m + 1)*Simp[A*(a^2 - b^2)*(m + 1) - a*(A*b - a*B + b*C)*(m + 1)*Csc[e + f*x
] + (A*b^2 - a*b*B + a^2*C)*(m + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a,
 b, e, f, A, B, C}, x] && NeQ[a^2 - b^2, 0] && LtQ[m, -1]
```

3.50. $\int \frac{1}{(a+b\csc(c+dx))^3}\,dx$

3.50.4 Maple [A] (verified)

Time = 1.05 (sec) , antiderivative size = 314, normalized size of antiderivative = 1.85

method	result
derivativedivides	$\frac{\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{a^3}-\frac{2b\left(\frac{\frac{4a^2b\left(4a^2-b^2\right)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^3}{8a^4-16a^2b^2+8b^4}+\frac{4a\left(10a^4+a^2b^2-2b^4\right)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2}{8a^4-16a^2b^2+8b^4}+\frac{4a^2b\left(16a^2-7b^2\right)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{8a^4-16a^2b^2+8b^4}}{\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2b+2a\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+b\right)^2}\right.}{a^3}}{d}$
default	$\frac{\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{a^3}-\frac{2b\left(\frac{\frac{4a^2b\left(4a^2-b^2\right)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^3}{8a^4-16a^2b^2+8b^4}+\frac{4a\left(10a^4+a^2b^2-2b^4\right)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2}{8a^4-16a^2b^2+8b^4}+\frac{4a^2b\left(16a^2-7b^2\right)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{8a^4-16a^2b^2+8b^4}}{\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^2b+2a\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+b\right)^2}\right.}{a^3}}{d}$
risch	$\frac{x}{a^3}-\frac{ib^2\left(7ia^3b\,e^{3i(dx+c)}-4iab^3e^{3i(dx+c)}-17ia^3b\,e^{i(dx+c)}+8iab^3e^{i(dx+c)}-6a^4e^{2i(dx+c)}-9a^2b^2e^{2i(dx+c)}+6b^4e^{2i(dx+c)}\right.}{\left(2b\,e^{i(dx+c)}-ia\,e^{2i(dx+c)}+ia\right)^2\left(-a^2+b^2\right)^2d\,a^3}$

input
```
int(1/(a+b*csc(d*x+c))^3,x,method=_RETURNVERBOSE)
```

output
```
1/d*(2/a^3*arctan(tan(1/2*d*x+1/2*c))-2/a^3*b*(4*(1/8*a^2*b*(4*a^2-b^2)/(a
^4-2*a^2*b^2+b^4)*tan(1/2*d*x+1/2*c)^3+1/8*a*(10*a^4+a^2*b^2-2*b^4)/(a^4-2
*a^2*b^2+b^4)*tan(1/2*d*x+1/2*c)^2+1/8*a^2*b*(16*a^2-7*b^2)/(a^4-2*a^2*b^2
+b^4)*tan(1/2*d*x+1/2*c)+1/8*a*b^2*(5*a^2-2*b^2)/(a^4-2*a^2*b^2+b^4))/(tan
(1/2*d*x+1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^2+2*(6*a^4-5*a^2*b^2+2*b^4)/
(4*a^4-8*a^2*b^2+4*b^4)/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(1/2*d*x+1/2*c
)+2*a)/(-a^2+b^2)^(1/2))))
```

3.50.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 433 vs. 2(161) = 322.

Time = 0.30 (sec) , antiderivative size = 933, normalized size of antiderivative = 5.49

$$\int \frac{1}{(a+b\csc(c+dx))^3}\,dx$$

$$=\left[\frac{4\left(a^8-3a^6b^2+3a^4b^4-a^2b^6\right)dx\cos\left(dx+c\right)^2-4\left(a^8-2a^6b^2+2a^2b^6-b^8\right)dx-\left(6a^6b+a^4b^3-3a^2b^5\right.}{}\right.$$

input
```
integrate(1/(a+b*csc(d*x+c))^3,x, algorithm="fricas")
```

output
```
[1/4*(4*(a^8 - 3*a^6*b^2 + 3*a^4*b^4 - a^2*b^6)*d*x*cos(d*x + c)^2 - 4*(a^
8 - 2*a^6*b^2 + 2*a^2*b^6 - b^8)*d*x - (6*a^6*b + a^4*b^3 - 3*a^2*b^5 + 2*
b^7 - (6*a^6*b - 5*a^4*b^3 + 2*a^2*b^5)*cos(d*x + c)^2 + 2*(6*a^5*b^2 - 5*
a^3*b^4 + 2*a*b^6)*sin(d*x + c))*sqrt(a^2 - b^2)*log(((a^2 - 2*b^2)*cos(d*
x + c)^2 + 2*a*b*sin(d*x + c) + a^2 + b^2 + 2*(b*cos(d*x + c)*sin(d*x + c)
 + a*cos(d*x + c))*sqrt(a^2 - b^2))/(a^2*cos(d*x + c)^2 - 2*a*b*sin(d*x +
c) - a^2 - b^2)) + 2*(5*a^5*b^3 - 7*a^3*b^5 + 2*a*b^7)*cos(d*x + c) - 2*(4
*(a^7*b - 3*a^5*b^3 + 3*a^3*b^5 - a*b^7)*d*x - 3*(2*a^6*b^2 - 3*a^4*b^4 +
a^2*b^6)*cos(d*x + c))*sin(d*x + c))/((a^11 - 3*a^9*b^2 + 3*a^7*b^4 - a^5*
b^6)*d*cos(d*x + c)^2 - 2*(a^10*b - 3*a^8*b^3 + 3*a^6*b^5 - a^4*b^7)*d*sin
(d*x + c) - (a^11 - 2*a^9*b^2 + 2*a^5*b^6 - a^3*b^8)*d), 1/2*(2*(a^8 - 3*a
^6*b^2 + 3*a^4*b^4 - a^2*b^6)*d*x*cos(d*x + c)^2 - 2*(a^8 - 2*a^6*b^2 + 2*
a^2*b^6 - b^8)*d*x - (6*a^6*b + a^4*b^3 - 3*a^2*b^5 + 2*b^7 - (6*a^6*b - 5
*a^4*b^3 + 2*a^2*b^5)*cos(d*x + c)^2 + 2*(6*a^5*b^2 - 5*a^3*b^4 + 2*a*b^6)
*sin(d*x + c))*sqrt(-a^2 + b^2)*arctan(-sqrt(-a^2 + b^2)*(b*sin(d*x + c) +
 a)/((a^2 - b^2)*cos(d*x + c))) + (5*a^5*b^3 - 7*a^3*b^5 + 2*a*b^7)*cos(d*
x + c) - (4*(a^7*b - 3*a^5*b^3 + 3*a^3*b^5 - a*b^7)*d*x - 3*(2*a^6*b^2 - 3
*a^4*b^4 + a^2*b^6)*cos(d*x + c))*sin(d*x + c))/((a^11 - 3*a^9*b^2 + 3*a^7
*b^4 - a^5*b^6)*d*cos(d*x + c)^2 - 2*(a^10*b - 3*a^8*b^3 + 3*a^6*b^5 - a^4
*b^7)*d*sin(d*x + c) - (a^11 - 2*a^9*b^2 + 2*a^5*b^6 - a^3*b^8)*d)]
```

3.50.6 Sympy [F]

$$\int \frac{1}{(a+b\csc(c+dx))^3}\,dx = \int \frac{1}{\left(a+b\csc\left(c+dx\right)\right)^3}\,dx$$

input
```
integrate(1/(a+b*csc(d*x+c))**3,x)
```

output
```
Integral((a + b*csc(c + d*x))**(-3), x)
```

3.50.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{1}{(a+b\csc(c+dx))^3}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(1/(a+b*csc(d*x+c))^3,x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.50.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 297, normalized size of antiderivative = 1.75

$$\int \frac{1}{(a+b\csc(c+dx))^3}\,dx = -\frac{\frac{(6\,a^4b-5\,a^2b^3+2\,b^5)\left(\pi\left\lfloor\frac{dx+c}{2\,\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+a}{\sqrt{-a^2+b^2}}\right)\right)}{(a^7-2\,a^5b^2+a^3b^4)\sqrt{-a^2+b^2}}+\frac{4\,a^3b^2\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^3-ab^4\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^3+10\,a^4b\tan\left(\frac{1}{2}\,dx\right.}{(a^6-2}}{d}$$

input
```
integrate(1/(a+b*csc(d*x+c))^3,x, algorithm="giac")
```

output
```
-((6*a^4*b - 5*a^2*b^3 + 2*b^5)*(pi*floor(1/2*(d*x + c)/pi + 1/2)*sgn(b) +
 arctan((b*tan(1/2*d*x + 1/2*c) + a)/sqrt(-a^2 + b^2)))/((a^7 - 2*a^5*b^2
+ a^3*b^4)*sqrt(-a^2 + b^2)) + (4*a^3*b^2*tan(1/2*d*x + 1/2*c)^3 - a*b^4*t
an(1/2*d*x + 1/2*c)^3 + 10*a^4*b*tan(1/2*d*x + 1/2*c)^2 + a^2*b^3*tan(1/2*
d*x + 1/2*c)^2 - 2*b^5*tan(1/2*d*x + 1/2*c)^2 + 16*a^3*b^2*tan(1/2*d*x + 1
/2*c) - 7*a*b^4*tan(1/2*d*x + 1/2*c) + 5*a^2*b^3 - 2*b^5)/((a^6 - 2*a^4*b^
2 + a^2*b^4)*(b*tan(1/2*d*x + 1/2*c)^2 + 2*a*tan(1/2*d*x + 1/2*c) + b)^2)
- (d*x + c)/a^3)/d
```

3.50.9 Mupad [B] (verification not implemented)

Time = 27.17 (sec) , antiderivative size = 5917, normalized size of antiderivative = 34.81

$$\int \frac{1}{(a+b\csc(c+dx))^3}\,dx = \text{Too large to display}$$

input
```
int(1/(a + b/sin(c + d*x))^3,x)
```

output
```
((2*b^5 - 5*a^2*b^3)/(a^2*(a^4 + b^4 - 2*a^2*b^2)) + (tan(c/2 + (d*x)/2)*(
7*b^4 - 16*a^2*b^2))/(a*(a^4 + b^4 - 2*a^2*b^2)) + (tan(c/2 + (d*x)/2)^3*(
b^4 - 4*a^2*b^2))/(a*(a^4 + b^4 - 2*a^2*b^2)) - (tan(c/2 + (d*x)/2)^2*(5*a
^2*b - 2*b^3)*(2*a^2 + b^2))/(a^2*(a^4 + b^4 - 2*a^2*b^2)))/(d*(tan(c/2 +
(d*x)/2)^2*(4*a^2 + 2*b^2) + b^2*tan(c/2 + (d*x)/2)^4 + b^2 + 4*a*b*tan(c/
2 + (d*x)/2)^3 + 4*a*b*tan(c/2 + (d*x)/2))) + (2*atan((((8*(4*a^2*b^10 - 1
6*a^4*b^8 + 24*a^6*b^6 - 16*a^8*b^4 + 4*a^10*b^2))/(a^13 + a^5*b^8 - 4*a^7
*b^6 + 6*a^9*b^4 - 4*a^11*b^2) - (((8*(4*a^14*b + 2*a^6*b^9 - 4*a^8*b^7 +
6*a^10*b^5 - 8*a^12*b^3))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11
*b^2) - (((8*(4*a^8*b^10 - 16*a^10*b^8 + 24*a^12*b^6 - 16*a^14*b^4 + 4*a^1
6*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4 - 4*a^11*b^2) + (8*tan(c/2
 + (d*x)/2)*(12*a^18*b - 8*a^8*b^11 + 44*a^10*b^9 - 96*a^12*b^7 + 104*a^14
*b^5 - 56*a^16*b^3))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4*a^12*b^2
))*1i)/a^3 + (8*tan(c/2 + (d*x)/2)*(8*a^6*b^10 - 36*a^8*b^8 + 72*a^10*b^6
- 68*a^12*b^4 + 24*a^14*b^2))/(a^14 + a^6*b^8 - 4*a^8*b^6 + 6*a^10*b^4 - 4
*a^12*b^2))*1i)/a^3 + (8*tan(c/2 + (d*x)/2)*(8*a^12*b - 8*a^2*b^11 + 44*a^
4*b^9 - 105*a^6*b^7 + 124*a^8*b^5 - 72*a^10*b^3))/(a^14 + a^6*b^8 - 4*a^8*
b^6 + 6*a^10*b^4 - 4*a^12*b^2))/a^3 + ((8*(4*a^2*b^10 - 16*a^4*b^8 + 24*a^
6*b^6 - 16*a^8*b^4 + 4*a^10*b^2))/(a^13 + a^5*b^8 - 4*a^7*b^6 + 6*a^9*b^4
- 4*a^11*b^2) + (((((8*(4*a^8*b^10 - 16*a^10*b^8 + 24*a^12*b^6 - 16*a^1...
```

3.51 $\int \frac{1}{(a+b\csc(c+dx))^4}\,dx$

3.51.1 Optimal result

Integrand size = 12, antiderivative size = 239

$$\int \frac{1}{(a+b\csc(c+dx))^4}\,dx = \frac{x}{a^4} + \frac{b(8a^6-8a^4b^2+7a^2b^4-2b^6)\,\text{arctanh}\left(\frac{a+b\tan\left(\frac{1}{2}(c+dx)\right)}{\sqrt{a^2-b^2}}\right)}{a^4\left(a^2-b^2\right)^{7/2}d} - \frac{b^2\cot(c+dx)}{3a\left(a^2-b^2\right)d(a+b\csc(c+dx))^3} - \frac{b^2(8a^2-3b^2)\cot(c+dx)}{6a^2\left(a^2-b^2\right)^2d(a+b\csc(c+dx))^2} - \frac{b^2(26a^4-17a^2b^2+6b^4)\cot(c+dx)}{6a^3\left(a^2-b^2\right)^3d(a+b\csc(c+dx))}$$

output
```
x/a^4+b*(8*a^6-8*a^4*b^2+7*a^2*b^4-2*b^6)*arctanh((a+b*tan(1/2*d*x+1/2*c))
/(a^2-b^2)^(1/2))/a^4/(a^2-b^2)^(7/2)/d-1/3*b^2*cot(d*x+c)/a/(a^2-b^2)/d/(
a+b*csc(d*x+c))^3-1/6*b^2*(8*a^2-3*b^2)*cot(d*x+c)/a^2/(a^2-b^2)^2/d/(a+b*
csc(d*x+c))^2-1/6*b^2*(26*a^4-17*a^2*b^2+6*b^4)*cot(d*x+c)/a^3/(a^2-b^2)^3
/d/(a+b*csc(d*x+c))
```

3.51.2 Mathematica [A] (verified)

Time = 5.56 (sec) , antiderivative size = 279, normalized size of antiderivative = 1.17

$$\int \frac{1}{(a+b\csc(c+dx))^4}\,dx$$
$$= \frac{\csc^3(c+dx)(b+a\sin(c+dx))\left(\frac{2ab^4\cot(c+dx)}{(-a+b)(a+b)}+\frac{ab^3(12a^2-7b^2)\cot(c+dx)(b+a\sin(c+dx))}{(a-b)^2(a+b)^2}-\frac{ab^2(36a^4-32a^2b^2+11b^4)\cot(c}{(a-b)^3(a+}\right.}{}$$

input
```
Integrate[(a + b*Csc[c + d*x])^(-4),x]
```

output
```
(Csc[c + d*x]^3*(b + a*Sin[c + d*x])*((2*a*b^4*Cot[c + d*x])/((-a + b)*(a
+ b)) + (a*b^3*(12*a^2 - 7*b^2)*Cot[c + d*x]*(b + a*Sin[c + d*x]))/((a - b
)^2*(a + b)^2) - (a*b^2*(36*a^4 - 32*a^2*b^2 + 11*b^4)*Cot[c + d*x]*(b + a
*Sin[c + d*x])^2)/((a - b)^3*(a + b)^3) + 6*(c + d*x)*Csc[c + d*x]*(b + a*
Sin[c + d*x])^3 - (6*b*(-8*a^6 + 8*a^4*b^2 - 7*a^2*b^4 + 2*b^6)*ArcTan[(a
+ b*Tan[(c + d*x)/2])/Sqrt[-a^2 + b^2]]*Csc[c + d*x]*(b + a*Sin[c + d*x])^
3)/(-a^2 + b^2)^(7/2)))/(6*a^4*d*(a + b*Csc[c + d*x])^4)
```

3.51.3 Rubi [A] (verified)

Time = 1.41 (sec) , antiderivative size = 314, normalized size of antiderivative = 1.31, number of steps used = 18, number of rules used = 17, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.417, Rules used = {3042, 4272, 25, 3042, 4548, 25, 3042, 4548, 27, 3042, 4407, 3042, 4318, 3042, 3139, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{(a+b\csc(c+dx))^4}\,dx$$
$$\Big\downarrow \; 3042$$
$$\int \frac{1}{(a+b\csc(c+dx))^4}dx$$
$$\Big\downarrow \; 4272$$

$$-\frac{\int -\frac{2b^2\csc^2(c+dx)-3ab\csc(c+dx)+3(a^2-b^2)}{(a+b\csc(c+dx))^3}dx}{3a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{3ad(a^2-b^2)(a+b\csc(c+dx))^3}$$

$\big\downarrow$ 25

$$\frac{\int \frac{2b^2\csc^2(c+dx)-3ab\csc(c+dx)+3(a^2-b^2)}{(a+b\csc(c+dx))^3}dx}{3a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{3ad(a^2-b^2)(a+b\csc(c+dx))^3}$$

$\big\downarrow$ 3042

$$\frac{\int \frac{2b^2\csc(c+dx)^2-3ab\csc(c+dx)+3(a^2-b^2)}{(a+b\csc(c+dx))^3}dx}{3a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{3ad(a^2-b^2)(a+b\csc(c+dx))^3}$$

$\big\downarrow$ 4548

$$\frac{-\frac{\int -\frac{6\left(a^2-b^2\right)^2+b^2\left(8a^2-3b^2\right)\csc^2(c+dx)-2ab\left(6a^2-b^2\right)\csc(c+dx)}{(a+b\csc(c+dx))^2}dx}{2a(a^2-b^2)}-\frac{b^2(8a^2-3b^2)\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}}{3a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{3ad(a^2-b^2)(a+b\csc(c+dx))^3}$$

$\big\downarrow$ 25

$$\frac{\frac{\int \frac{6\left(a^2-b^2\right)^2+b^2\left(8a^2-3b^2\right)\csc^2(c+dx)-2ab\left(6a^2-b^2\right)\csc(c+dx)}{(a+b\csc(c+dx))^2}dx}{2a(a^2-b^2)}-\frac{b^2(8a^2-3b^2)\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}}{3a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{3ad(a^2-b^2)(a+b\csc(c+dx))^3}$$

$\big\downarrow$ 3042

$$\frac{\frac{\int \frac{6\left(a^2-b^2\right)^2+b^2\left(8a^2-3b^2\right)\csc(c+dx)^2-2ab\left(6a^2-b^2\right)\csc(c+dx)}{(a+b\csc(c+dx))^2}dx}{2a(a^2-b^2)}-\frac{b^2(8a^2-3b^2)\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}}{3a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{3ad(a^2-b^2)(a+b\csc(c+dx))^3}$$

$\big\downarrow$ 4548

$$\frac{\frac{-\frac{\int -\frac{3\left(2\left(a^2-b^2\right)^3-ab\left(6a^4-2b^2a^2+b^4\right)\csc(c+dx)\right)}{a+b\csc(c+dx)}dx}{a\left(a^2-b^2\right)}-\frac{b^2\left(26a^4-17a^2b^2+6b^4\right)\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2(8a^2-3b^2)\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}}{3a(a^2-b^2)}-\frac{b^2\cot(c+dx)}{3ad(a^2-b^2)(a+b\csc(c+dx))^3}$$

$\big\downarrow$ 27

3.51. $\int \frac{1}{(a+b\csc(c+dx))^4}\,dx$

$$\frac{\frac{\frac{3\int \frac{2\left(a^2-b^2\right)^3-ab\left(6a^4-2b^2a^2+b^4\right)\csc(c+dx)}{a+b\csc(c+dx)}dx}{a\left(a^2-b^2\right)}-\frac{b^2\left(26a^4-17a^2b^2+6b^4\right)\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2(8a^2-3b^2)\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}}{3a\left(a^2-b^2\right)}-\frac{b^2\cot(c+dx)}{3ad\left(a^2-b^2\right)(a+b\csc(c+dx))^3}$$

$\Big\downarrow$ 3042

$$\frac{\frac{\frac{3\int \frac{2\left(a^2-b^2\right)^3-ab\left(6a^4-2b^2a^2+b^4\right)\csc(c+dx)}{a+b\csc(c+dx)}dx}{a\left(a^2-b^2\right)}-\frac{b^2\left(26a^4-17a^2b^2+6b^4\right)\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2(8a^2-3b^2)\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}}{3a\left(a^2-b^2\right)}-\frac{b^2\cot(c+dx)}{3ad\left(a^2-b^2\right)(a+b\csc(c+dx))^3}$$

$\Big\downarrow$ 4407

$$\frac{\frac{\frac{3\left(\frac{2x\left(a^2-b^2\right)^3}{a}-\frac{b\left(8a^6-8a^4b^2+7a^2b^4-2b^6\right)\int \frac{\csc(c+dx)}{a+b\csc(c+dx)}dx}{a}\right)}{a\left(a^2-b^2\right)}-\frac{b^2\left(26a^4-17a^2b^2+6b^4\right)\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2(8a^2-3b^2)\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}}{3a\left(a^2-b^2\right)}-\frac{b^2\cot(c+dx)}{3ad\left(a^2-b^2\right)(a+b\csc(c+dx))^3}$$

$\Big\downarrow$ 3042

$$\frac{\frac{\frac{3\left(\frac{2x\left(a^2-b^2\right)^3}{a}-\frac{b\left(8a^6-8a^4b^2+7a^2b^4-2b^6\right)\int \frac{\csc(c+dx)}{a+b\csc(c+dx)}dx}{a}\right)}{a\left(a^2-b^2\right)}-\frac{b^2\left(26a^4-17a^2b^2+6b^4\right)\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2(8a^2-3b^2)\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}}{3a\left(a^2-b^2\right)}-\frac{b^2\cot(c+dx)}{3ad\left(a^2-b^2\right)(a+b\csc(c+dx))^3}$$

$\Big\downarrow$ 4318

$$\frac{\frac{\frac{3\left(\frac{2x\left(a^2-b^2\right)^3}{a}-\frac{\left(8a^6-8a^4b^2+7a^2b^4-2b^6\right)\int \frac{1}{\frac{a\sin(c+dx)}{b}+1}dx}{a}\right)}{a\left(a^2-b^2\right)}-\frac{b^2\left(26a^4-17a^2b^2+6b^4\right)\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2(8a^2-3b^2)\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}}{3a\left(a^2-b^2\right)}-\frac{b^2\cot(c+dx)}{3ad\left(a^2-b^2\right)(a+b\csc(c+dx))^3}$$

$\Big\downarrow$ 3042

3.51. $\int \frac{1}{(a+b\csc(c+dx))^4}\,dx$

$$\frac{\frac{3\left(\frac{2x\left(a^2-b^2\right)^3}{a}-\frac{\left(8a^6-8a^4b^2+7a^2b^4-2b^6\right)\int \frac{1}{\frac{a\sin(c+dx)}{b}+1}dx}{a}\right)}{a\left(a^2-b^2\right)}-\frac{b^2\left(26a^4-17a^2b^2+6b^4\right)\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\left(8a^2-3b^2\right)\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx))^2}}{3a\left(a^2-b^2\right)} - \frac{b^2\cot(c+dx)}{3ad\left(a^2-b^2\right)(a+b\csc(c+dx))^3}$$

$\downarrow$ 3139

$$\frac{\frac{3\left(\frac{2x\left(a^2-b^2\right)^3}{a}-\frac{2\left(8a^6-8a^4b^2+7a^2b^4-2b^6\right)\int \frac{1}{\tan^2\left(\frac{1}{2}(c+dx)\right)+\frac{2a\tan\left(\frac{1}{2}(c+dx)\right)}{b}+1}d\tan\left(\frac{1}{2}(c+dx)\right)}{ad}\right)}{a\left(a^2-b^2\right)}-\frac{b^2\left(26a^4-17a^2b^2+6b^4\right)\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2(8a^2-}{2ad(a^2-b^2)}}{3a\left(a^2-b^2\right)}$$
$$\frac{b^2\cot(c+dx)}{3ad\left(a^2-b^2\right)(a+b\csc(c+dx))^3}$$

$\downarrow$ 1083

$$\frac{\frac{3\left(\frac{4\left(8a^6-8a^4b^2+7a^2b^4-2b^6\right)\int \frac{1}{-\left(\frac{2a}{b}+2\tan\left(\frac{1}{2}(c+dx)\right)\right)^2-4\left(1-\frac{a^2}{b^2}\right)}d\left(\frac{2a}{b}+2\tan\left(\frac{1}{2}(c+dx)\right)\right)}{ad}+\frac{2x\left(a^2-b^2\right)^3}{a}\right)}{a\left(a^2-b^2\right)}-\frac{b^2\left(26a^4-17a^2b^2+6b^4\right)\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b}{2ad}}{3a\left(a^2-b^2\right)}$$
$$\frac{b^2\cot(c+dx)}{3ad\left(a^2-b^2\right)(a+b\csc(c+dx))^3}$$

$\downarrow$ 219

$$\frac{\frac{3\left(\frac{2x\left(a^2-b^2\right)^3}{a}+\frac{2b\left(8a^6-8a^4b^2+7a^2b^4-2b^6\right)\text{arctanh}\left(\frac{b\left(\frac{2a}{b}+2\tan\left(\frac{1}{2}(c+dx)\right)\right)}{2\sqrt{a^2-b^2}}\right)}{ad\sqrt{a^2-b^2}}\right)}{a\left(a^2-b^2\right)}-\frac{b^2\left(26a^4-17a^2b^2+6b^4\right)\cot(c+dx)}{ad\left(a^2-b^2\right)(a+b\csc(c+dx))}}{2a(a^2-b^2)}-\frac{b^2\left(8a^2-3b^2\right)\cot(c+dx)}{2ad(a^2-b^2)(a+b\csc(c+dx}}{3a\left(a^2-b^2\right)}$$
$$\frac{b^2\cot(c+dx)}{3ad\left(a^2-b^2\right)(a+b\csc(c+dx))^3}$$

input

```
Int[(a + b*Csc[c + d*x])^(-4),x]
```

3.51. $\int \frac{1}{(a+b\csc(c+dx))^4}\,dx$

output
```
-1/3*(b^2*Cot[c + d*x])/(a*(a^2 - b^2)*d*(a + b*Csc[c + d*x])^3) + (-1/2*(
b^2*(8*a^2 - 3*b^2)*Cot[c + d*x])/(a*(a^2 - b^2)*d*(a + b*Csc[c + d*x])^2)
 + ((3*((2*(a^2 - b^2)^3*x)/a + (2*b*(8*a^6 - 8*a^4*b^2 + 7*a^2*b^4 - 2*b^
6)*ArcTanh[(b*((2*a)/b + 2*Tan[(c + d*x)/2]))/(2*Sqrt[a^2 - b^2])])/(a*Sqr
t[a^2 - b^2]*d)))/(a*(a^2 - b^2)) - (b^2*(26*a^4 - 17*a^2*b^2 + 6*b^4)*Cot
[c + d*x])/(a*(a^2 - b^2)*d*(a + b*Csc[c + d*x])))/(2*a*(a^2 - b^2)))/(3*a
*(a^2 - b^2))
```

3.51.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4272
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Simp[b^2*Cot[
c + d*x]*((a + b*Csc[c + d*x])^(n + 1)/(a*d*(n + 1)*(a^2 - b^2))), x] + Sim
p[1/(a*(n + 1)*(a^2 - b^2))   Int[(a + b*Csc[c + d*x])^(n + 1)*Simp[(a^2 -
b^2)*(n + 1) - a*b*(n + 1)*Csc[c + d*x] + b^2*(n + 2)*Csc[c + d*x]^2, x], x
], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && LtQ[n, -1] && Integ
erQ[2*n]
```

rule 4318
```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)), x_Symbo
l] :> Simp[1/b   Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

rule 4407
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
 (a_)), x_Symbol] :> Simp[c*(x/a), x] - Simp[(b*c - a*d)/a   Int[Csc[e + f*
x]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
- a*d, 0]
```

rule 4548
```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_Symbol] :> Simp[(A*b^2 -
a*b*B + a^2*C)*Cot[e + f*x]*((a + b*Csc[e + f*x])^(m + 1)/(a*f*(m + 1)*(a^2
 - b^2))), x] + Simp[1/(a*(m + 1)*(a^2 - b^2))   Int[(a + b*Csc[e + f*x])^(
m + 1)*Simp[A*(a^2 - b^2)*(m + 1) - a*(A*b - a*B + b*C)*(m + 1)*Csc[e + f*x
] + (A*b^2 - a*b*B + a^2*C)*(m + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a,
 b, e, f, A, B, C}, x] && NeQ[a^2 - b^2, 0] && LtQ[m, -1]
```

3.51.4 Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 544 vs. $2(228) = 456$.

Time = 2.09 (sec) , antiderivative size = 545, normalized size of antiderivative = 2.28

method	result
derivativedivides	$\frac{\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{a^4}-\frac{2b\left(\frac{8b^2a^2\left(6a^4-2a^2b^2+b^4\right)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^5}{16a^6-48a^4b^2+48a^2b^4-16b^6}+\frac{8ab\left(28a^6-4a^4b^2-a^2b^4+2b^6\right)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^4}{16a^6-48a^4b^2+48a^2b^4-16b^6}+\frac{8a^2\left(52a^6+44\right.}{24a}\right.}{}}{}$
default	$\frac{\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{a^4}-\frac{2b\left(\frac{8b^2a^2\left(6a^4-2a^2b^2+b^4\right)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^5}{16a^6-48a^4b^2+48a^2b^4-16b^6}+\frac{8ab\left(28a^6-4a^4b^2-a^2b^4+2b^6\right)\tan\left(\frac{dx}{2}+\frac{c}{2}\right)^4}{16a^6-48a^4b^2+48a^2b^4-16b^6}+\frac{8a^2\left(52a^6+44\right.}{24a}\right.}{}}{}$
risch	$\frac{x}{a^4}-\frac{ib^2\left(-36ia^7+32ia^5b^2-132ia^5b^2e^{4i(dx+c)}-204ia^3b^4e^{2i(dx+c)}-48b\,a^6e^{5i(dx+c)}+51b^3a^4e^{5i(dx+c)}-18b^5a^2e^{5i(dx+\right.}{}$

input
```
int(1/(a+b*csc(d*x+c))^4,x,method=_RETURNVERBOSE)
```

output
```
1/d*(2/a^4*arctan(tan(1/2*d*x+1/2*c))-2/a^4*b*(8*(1/16*b^2*a^2*(6*a^4-2*a^
2*b^2+b^4)/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^5+1/16*a*b*(28
*a^6-4*a^4*b^2-a^2*b^4+2*b^6)/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/
2*c)^4+1/24*a^2*(52*a^6+44*a^4*b^2-39*a^2*b^4+18*b^6)/(a^6-3*a^4*b^2+3*a^2
*b^4-b^6)*tan(1/2*d*x+1/2*c)^3+1/8*a*b*(38*a^6-19*a^4*b^2+4*a^2*b^4+2*b^6)
/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)^2+1/16*(46*a^4-32*a^2*b^
2+11*b^4)*a^2*b^2/(a^6-3*a^4*b^2+3*a^2*b^4-b^6)*tan(1/2*d*x+1/2*c)+1/48*a*
b^3*(26*a^4-17*a^2*b^2+6*b^4)/(a^6-3*a^4*b^2+3*a^2*b^4-b^6))/(tan(1/2*d*x+
1/2*c)^2*b+2*a*tan(1/2*d*x+1/2*c)+b)^3+4*(8*a^6-8*a^4*b^2+7*a^2*b^4-2*b^6)
/(8*a^6-24*a^4*b^2+24*a^2*b^4-8*b^6)/(-a^2+b^2)^(1/2)*arctan(1/2*(2*b*tan(
1/2*d*x+1/2*c)+2*a)/(-a^2+b^2)^(1/2))))
```

3.51.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 744 vs. $2(228) = 456$.

Time = 0.34 (sec) , antiderivative size = 1554, normalized size of antiderivative = 6.50

$$\int \frac{1}{(a+b\csc(c+dx))^4}\,dx = \text{Too large to display}$$

input
```
integrate(1/(a+b*csc(d*x+c))^4,x, algorithm="fricas")
```

output
```
[1/12*(36*(a^10*b - 4*a^8*b^3 + 6*a^6*b^5 - 4*a^4*b^7 + a^2*b^9)*d*x*cos(d
*x + c)^2 - 2*(36*a^9*b^2 - 68*a^7*b^4 + 43*a^5*b^6 - 11*a^3*b^8)*cos(d*x
+ c)^3 - 12*(3*a^10*b - 11*a^8*b^3 + 14*a^6*b^5 - 6*a^4*b^7 - a^2*b^9 + b^
11)*d*x - 3*(24*a^8*b^2 - 16*a^6*b^4 + 13*a^4*b^6 + a^2*b^8 - 2*b^10 - 3*(
8*a^8*b^2 - 8*a^6*b^4 + 7*a^4*b^6 - 2*a^2*b^8)*cos(d*x + c)^2 + (8*a^9*b +
 16*a^7*b^3 - 17*a^5*b^5 + 19*a^3*b^7 - 6*a*b^9 - (8*a^9*b - 8*a^7*b^3 + 7
*a^5*b^5 - 2*a^3*b^7)*cos(d*x + c)^2)*sin(d*x + c))*sqrt(a^2 - b^2)*log(((
a^2 - 2*b^2)*cos(d*x + c)^2 + 2*a*b*sin(d*x + c) + a^2 + b^2 + 2*(b*cos(d*
x + c)*sin(d*x + c) + a*cos(d*x + c))*sqrt(a^2 - b^2))/(a^2*cos(d*x + c)^2
 - 2*a*b*sin(d*x + c) - a^2 - b^2)) + 12*(6*a^9*b^2 - 7*a^7*b^4 + 2*a^3*b^
8 - a*b^10)*cos(d*x + c) + 6*(2*(a^11 - 4*a^9*b^2 + 6*a^7*b^4 - 4*a^5*b^6
+ a^3*b^8)*d*x*cos(d*x + c)^2 - 2*(a^11 - a^9*b^2 - 6*a^7*b^4 + 14*a^5*b^6
 - 11*a^3*b^8 + 3*a*b^10)*d*x + 5*(4*a^8*b^3 - 7*a^6*b^5 + 4*a^4*b^7 - a^2
*b^9)*cos(d*x + c))*sin(d*x + c))/(3*(a^14*b - 4*a^12*b^3 + 6*a^10*b^5 - 4
*a^8*b^7 + a^6*b^9)*d*cos(d*x + c)^2 - (3*a^14*b - 11*a^12*b^3 + 14*a^10*b
^5 - 6*a^8*b^7 - a^6*b^9 + a^4*b^11)*d + ((a^15 - 4*a^13*b^2 + 6*a^11*b^4
- 4*a^9*b^6 + a^7*b^8)*d*cos(d*x + c)^2 - (a^15 - a^13*b^2 - 6*a^11*b^4 +
14*a^9*b^6 - 11*a^7*b^8 + 3*a^5*b^10)*d)*sin(d*x + c)), 1/6*(18*(a^10*b -
4*a^8*b^3 + 6*a^6*b^5 - 4*a^4*b^7 + a^2*b^9)*d*x*cos(d*x + c)^2 - (36*a^9*
b^2 - 68*a^7*b^4 + 43*a^5*b^6 - 11*a^3*b^8)*cos(d*x + c)^3 - 6*(3*a^10*...
```

3.51.6 Sympy [F]

$$\int \frac{1}{(a+b\csc(c+dx))^4}\,dx = \int \frac{1}{\left(a+b\csc\left(c+dx\right)\right)^4}\,dx$$

input
```
integrate(1/(a+b*csc(d*x+c))**4,x)
```

output
```
Integral((a + b*csc(c + d*x))**(-4), x)
```

3.51.7 Maxima [F(-2)]

Exception generated.

$$\int \frac{1}{(a+b\csc(c+dx))^4}\,dx = \text{Exception raised: ValueError}$$

input
```
integrate(1/(a+b*csc(d*x+c))^4,x, algorithm="maxima")
```

output
```
Exception raised: ValueError >> Computation failed since Maxima requested
additional constraints; using the 'assume' command before evaluation *may*
 help (example of legal syntax is 'assume(4*a^2-4*b^2>0)', see `assume?` f
or more de
```

3.51.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 535 vs. $2(228) = 456$.

Time = 0.30 (sec) , antiderivative size = 535, normalized size of antiderivative = 2.24

$$\int \frac{1}{(a+b\csc(c+dx))^4}\,dx =$$

$$-\frac{\frac{3\,(8\,a^6b-8\,a^4b^3+7\,a^2b^5-2\,b^7)\left(\pi\left\lfloor\frac{dx+c}{2\,\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(b)+\arctan\left(\frac{b\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+a}{\sqrt{-a^2+b^2}}\right)\right)}{(a^{10}-3\,a^8b^2+3\,a^6b^4-a^4b^6)\sqrt{-a^2+b^2}}+\frac{18\,a^5b^3\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^5-6\,a^3b^5\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)^5+3\,a}{}}{}$$

input
```
integrate(1/(a+b*csc(d*x+c))^4,x, algorithm="giac")
```

output
```
-1/3*(3*(8*a^6*b - 8*a^4*b^3 + 7*a^2*b^5 - 2*b^7)*(pi*floor(1/2*(d*x + c)/
pi + 1/2)*sgn(b) + arctan((b*tan(1/2*d*x + 1/2*c) + a)/sqrt(-a^2 + b^2)))/
((a^10 - 3*a^8*b^2 + 3*a^6*b^4 - a^4*b^6)*sqrt(-a^2 + b^2)) + (18*a^5*b^3*
tan(1/2*d*x + 1/2*c)^5 - 6*a^3*b^5*tan(1/2*d*x + 1/2*c)^5 + 3*a*b^7*tan(1/
2*d*x + 1/2*c)^5 + 84*a^6*b^2*tan(1/2*d*x + 1/2*c)^4 - 12*a^4*b^4*tan(1/2*
d*x + 1/2*c)^4 - 3*a^2*b^6*tan(1/2*d*x + 1/2*c)^4 + 6*b^8*tan(1/2*d*x + 1/
2*c)^4 + 104*a^7*b*tan(1/2*d*x + 1/2*c)^3 + 88*a^5*b^3*tan(1/2*d*x + 1/2*c
)^3 - 78*a^3*b^5*tan(1/2*d*x + 1/2*c)^3 + 36*a*b^7*tan(1/2*d*x + 1/2*c)^3
+ 228*a^6*b^2*tan(1/2*d*x + 1/2*c)^2 - 114*a^4*b^4*tan(1/2*d*x + 1/2*c)^2
+ 24*a^2*b^6*tan(1/2*d*x + 1/2*c)^2 + 12*b^8*tan(1/2*d*x + 1/2*c)^2 + 138*
a^5*b^3*tan(1/2*d*x + 1/2*c) - 96*a^3*b^5*tan(1/2*d*x + 1/2*c) + 33*a*b^7*
tan(1/2*d*x + 1/2*c) + 26*a^4*b^4 - 17*a^2*b^6 + 6*b^8)/((a^9 - 3*a^7*b^2
+ 3*a^5*b^4 - a^3*b^6)*(b*tan(1/2*d*x + 1/2*c)^2 + 2*a*tan(1/2*d*x + 1/2*c
) + b)^3) - 3*(d*x + c)/a^4)/d
```

3.51.9 Mupad [B] (verification not implemented)

Time = 32.00 (sec) , antiderivative size = 8167, normalized size of antiderivative = 34.17

$$\int \frac{1}{(a+b\csc(c+dx))^4}\,dx = \text{Too large to display}$$

input
```
int(1/(a + b/sin(c + d*x))^4,x)
```

output

```
(2*atan((((8*(4*a^3*b^14 - 24*a^5*b^12 + 60*a^7*b^10 - 80*a^9*b^8 + 60*a^1
1*b^6 - 24*a^13*b^4 + 4*a^15*b^2))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^1
2*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) - (((8*(4*a^20*b + 2*a^8*b
^13 - 14*a^10*b^11 + 30*a^12*b^9 - 30*a^14*b^7 + 20*a^16*b^5 - 12*a^18*b^3
))/(a^20 + a^8*b^12 - 6*a^10*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^
4 - 6*a^18*b^2) - (((8*(4*a^11*b^14 - 24*a^13*b^12 + 60*a^15*b^10 - 80*a^1
7*b^8 + 60*a^19*b^6 - 24*a^21*b^4 + 4*a^23*b^2))/(a^20 + a^8*b^12 - 6*a^10
*b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c/2
 + (d*x)/2)*(12*a^25*b - 8*a^11*b^15 + 60*a^13*b^13 - 192*a^15*b^11 + 340*
a^17*b^9 - 360*a^19*b^7 + 228*a^21*b^5 - 80*a^23*b^3))/(a^21 + a^9*b^12 -
6*a^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*1i)/a
^4 + (8*tan(c/2 + (d*x)/2)*(8*a^8*b^14 - 52*a^10*b^12 + 140*a^12*b^10 - 22
0*a^14*b^8 + 220*a^16*b^6 - 128*a^18*b^4 + 32*a^20*b^2))/(a^21 + a^9*b^12
- 6*a^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^19*b^2))*1i)
/a^4 + (8*tan(c/2 + (d*x)/2)*(8*a^17*b - 8*a^3*b^15 + 60*a^5*b^13 - 189*a^
7*b^11 + 344*a^9*b^9 - 396*a^11*b^7 + 272*a^13*b^5 - 116*a^15*b^3))/(a^21
+ a^9*b^12 - 6*a^11*b^10 + 15*a^13*b^8 - 20*a^15*b^6 + 15*a^17*b^4 - 6*a^1
9*b^2))/a^4 + ((((((8*(4*a^11*b^14 - 24*a^13*b^12 + 60*a^15*b^10 - 80*a^17
*b^8 + 60*a^19*b^6 - 24*a^21*b^4 + 4*a^23*b^2))/(a^20 + a^8*b^12 - 6*a^10*
b^10 + 15*a^12*b^8 - 20*a^14*b^6 + 15*a^16*b^4 - 6*a^18*b^2) + (8*tan(c...
```

3.51. $\int \frac{1}{(a+b\csc(c+dx))^4}\,dx$

3.52 $\int \frac{1}{3+5\csc(c+dx)}\,dx$

3.52.1 Optimal result

Integrand size = 12, antiderivative size = 31

$$\int \frac{1}{3+5\csc(c+dx)}\,dx = -\frac{x}{12}-\frac{5\arctan\left(\frac{\cos(c+dx)}{3+\sin(c+dx)}\right)}{6d}$$

output
```
-1/12*x-5/6*arctan(cos(d*x+c)/(3+sin(d*x+c)))/d
```

3.52.2 Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 66 vs. 2(31) = 62.

Time = 0.16 (sec) , antiderivative size = 66, normalized size of antiderivative = 2.13

$$\int \frac{1}{3+5\csc(c+dx)}\,dx = \frac{2(c+dx)-5\arctan\left(\frac{2\left(\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)\right)}{\cos\left(\frac{1}{2}(c+dx)\right)-\sin\left(\frac{1}{2}(c+dx)\right)}\right)}{6d}$$

input
```
Integrate[(3 + 5*Csc[c + d*x])^(-1),x]
```

output
```
(2*(c + d*x) - 5*ArcTan[(2*(Cos[(c + d*x)/2] + Sin[(c + d*x)/2]))/(Cos[(c
+ d*x)/2] - Sin[(c + d*x)/2])])/(6*d)
```

3.52.3 Rubi [A] (verified)

Time = 0.22 (sec) , antiderivative size = 41, normalized size of antiderivative = 1.32, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.333, Rules used = {3042, 4270, 3042, 3136}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{5\csc(c+dx)+3}dx$$
$$\Big\downarrow \text{3042}$$
$$\int \frac{1}{5\csc(c+dx)+3}dx$$
$$\Big\downarrow \text{4270}$$
$$\frac{x}{3}-\frac{1}{3}\int \frac{1}{\frac{3}{5}\sin(c+dx)+1}dx$$
$$\Big\downarrow \text{3042}$$
$$\frac{x}{3}-\frac{1}{3}\int \frac{1}{\frac{3}{5}\sin(c+dx)+1}dx$$
$$\Big\downarrow \text{3136}$$
$$\frac{1}{3}\left(-\frac{5\arctan\left(\frac{\cos(c+dx)}{\sin(c+dx)+3}\right)}{2d}-\frac{5x}{4}\right)+\frac{x}{3}$$

input
```
Int[(3 + 5*Csc[c + d*x])^(-1),x]
```

output
```
x/3 + ((-5*x)/4 - (5*ArcTan[Cos[c + d*x]/(3 + Sin[c + d*x])])/(2*d))/3
```

3.52.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3136
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{q = Rt[
a^2 - b^2, 2]}, Simp[x/q, x] + Simp[(2/(d*q))*ArcTan[b*(Cos[c + d*x]/(a + q
 + b*Sin[c + d*x]))], x]] /; FreeQ[{a, b, c, d}, x] && GtQ[a^2 - b^2, 0] &&
 PosQ[a]
```

rule 4270
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(-1), x_Symbol] :> Simp[x/a, x]
- Simp[1/a   Int[1/(1 + (a/b)*Sin[c + d*x]), x], x] /; FreeQ[{a, b, c, d},
x] && NeQ[a^2 - b^2, 0]
```

3.52.4 Maple [A] (verified)

Time = 0.56 (sec) , antiderivative size = 34, normalized size of antiderivative = 1.10

method	result	size
derivativedivides	$\frac{\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{3}-\frac{5\arctan\left(\frac{5\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{4}+\frac{3}{4}\right)}{6}}{d}$	34
default	$\frac{\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{3}-\frac{5\arctan\left(\frac{5\tan\left(\frac{dx}{2}+\frac{c}{2}\right)}{4}+\frac{3}{4}\right)}{6}}{d}$	34
risch	$\frac{x}{3}-\frac{5i\ln\left(e^{i(dx+c)}+3i\right)}{12d}+\frac{5i\ln\left(e^{i(dx+c)}+\frac{i}{3}\right)}{12d}$	43
parallelrisch	$\frac{5i\ln\left(5\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+3-4i\right)-5i\ln\left(5\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+3+4i\right)+4dx}{12d}$	49

input
```
int(1/(3+5*csc(d*x+c)),x,method=_RETURNVERBOSE)
```

output
```
1/d*(2/3*arctan(tan(1/2*d*x+1/2*c))-5/6*arctan(5/4*tan(1/2*d*x+1/2*c)+3/4)
)
```

3.52.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 33, normalized size of antiderivative = 1.06

$$\int \frac{1}{3+5\csc(c+dx)}\,dx=\frac{4\,dx-5\,\arctan\left(\frac{5\,\sin(dx+c)+3}{4\,\cos(dx+c)}\right)}{12\,d}$$

input
```
integrate(1/(3+5*csc(d*x+c)),x, algorithm="fricas")
```

output
```
1/12*(4*d*x - 5*arctan(1/4*(5*sin(d*x + c) + 3)/cos(d*x + c)))/d
```

3.52.6 Sympy [F]

$$\int \frac{1}{3+5\csc(c+dx)}\,dx = \int \frac{1}{5\csc(c+dx)+3}\,dx$$

input
```
integrate(1/(3+5*csc(d*x+c)),x)
```

output
```
Integral(1/(5*csc(c + d*x) + 3), x)
```

3.52.7 Maxima [A] (verification not implemented)

Time = 0.33 (sec) , antiderivative size = 49, normalized size of antiderivative = 1.58

$$\int \frac{1}{3+5\csc(c+dx)}\,dx = -\frac{5\arctan\left(\frac{5\sin(dx+c)}{4(\cos(dx+c)+1)}+\frac{3}{4}\right)-4\arctan\left(\frac{\sin(dx+c)}{\cos(dx+c)+1}\right)}{6d}$$

input
```
integrate(1/(3+5*csc(d*x+c)),x, algorithm="maxima")
```

output
```
-1/6*(5*arctan(5/4*sin(d*x + c)/(cos(d*x + c) + 1) + 3/4) - 4*arctan(sin(d
*x + c)/(cos(d*x + c) + 1)))/d
```

3.52.8 Giac [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 49, normalized size of antiderivative = 1.58

$$\int \frac{1}{3+5\csc(c+dx)}\,dx = -\frac{dx+c+10\arctan\left(-\frac{3\cos(dx+c)+\sin(dx+c)+3}{\cos(dx+c)-3\sin(dx+c)-9}\right)}{12d}$$

input
```
integrate(1/(3+5*csc(d*x+c)),x, algorithm="giac")
```

output
```
-1/12*(d*x + c + 10*arctan(-(3*cos(d*x + c) + sin(d*x + c) + 3)/(cos(d*x +
 c) - 3*sin(d*x + c) - 9)))/d
```

3.52.9 Mupad [B] (verification not implemented)

Time = 19.32 (sec) , antiderivative size = 39, normalized size of antiderivative = 1.26

$$\int \frac{1}{3+5\csc(c+dx)}\,dx = \frac{x}{3} - \frac{5\,\mathrm{atan}\left(\frac{7\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)-15}{24\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)+20}\right)}{6\,d}$$

input
```
int(1/(5/sin(c + d*x) + 3),x)
```

output
```
x/3 - (5*atan((7*tan(c/2 + (d*x)/2) - 15)/(24*tan(c/2 + (d*x)/2) + 20)))/(
6*d)
```

3.53 $\int \frac{1}{5+3\csc(c+dx)}\,dx$

3.53.1 Optimal result

Integrand size = 12, antiderivative size = 68

$$\int \frac{1}{5+3\csc(c+dx)}\,dx = \frac{x}{5}+\frac{3\log\left(3\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)\right)}{20d} - \frac{3\log\left(\cos\left(\frac{1}{2}(c+dx)\right)+3\sin\left(\frac{1}{2}(c+dx)\right)\right)}{20d}$$

output
```
1/5*x+3/20*ln(3*cos(1/2*d*x+1/2*c)+sin(1/2*d*x+1/2*c))/d-3/20*ln(cos(1/2*d
*x+1/2*c)+3*sin(1/2*d*x+1/2*c))/d
```

3.53.2 Mathematica [A] (verified)

Time = 0.15 (sec) , antiderivative size = 67, normalized size of antiderivative = 0.99

$$\int \frac{1}{5+3\csc(c+dx)}\,dx = \frac{4(c+dx)+3\log\left(3\cos\left(\frac{1}{2}(c+dx)\right)+\sin\left(\frac{1}{2}(c+dx)\right)\right)-3\log\left(\cos\left(\frac{1}{2}(c+dx)\right)+3\sin\left(\frac{1}{2}(c+dx)\right)\right)}{20d}$$

input
```
Integrate[(5 + 3*Csc[c + d*x])^(-1),x]
```

output
```
(4*(c + d*x) + 3*Log[3*Cos[(c + d*x)/2] + Sin[(c + d*x)/2]] - 3*Log[Cos[(c
 + d*x)/2] + 3*Sin[(c + d*x)/2]])/(20*d)
```

3.53.3 Rubi [A] (verified)

Time = 0.26 (sec) , antiderivative size = 50, normalized size of antiderivative = 0.74, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.500, Rules used = {3042, 4270, 3042, 3139, 1081, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{3\csc(c+dx)+5}dx$$

$$\Big\downarrow \ 3042$$

$$\int \frac{1}{3\csc(c+dx)+5}dx$$

$$\Big\downarrow \ 4270$$

$$\frac{x}{5}-\frac{1}{5}\int \frac{1}{\frac{5}{3}\sin(c+dx)+1}dx$$

$$\Big\downarrow \ 3042$$

$$\frac{x}{5}-\frac{1}{5}\int \frac{1}{\frac{5}{3}\sin(c+dx)+1}dx$$

$$\Big\downarrow \ 3139$$

$$\frac{x}{5}-\frac{2\int \frac{1}{\tan^2\left(\frac{1}{2}(c+dx)\right)+\frac{10}{3}\tan\left(\frac{1}{2}(c+dx)\right)+1}d\tan\left(\frac{1}{2}(c+dx)\right)}{5d}$$

$$\Big\downarrow \ 1081$$

$$\frac{x}{5}-\frac{2\int \left(\frac{9}{8\left(3\tan\left(\frac{1}{2}(c+dx)\right)+1\right)}-\frac{3}{8\left(\tan\left(\frac{1}{2}(c+dx)\right)+3\right)}\right)d\tan\left(\frac{1}{2}(c+dx)\right)}{5d}$$

$$\Big\downarrow \ 2009$$

$$\frac{x}{5}-\frac{2\left(\frac{3}{8}\log\left(3\tan\left(\frac{1}{2}(c+dx)\right)+1\right)-\frac{3}{8}\log\left(\tan\left(\frac{1}{2}(c+dx)\right)+3\right)\right)}{5d}$$

input
```
Int[(5 + 3*Csc[c + d*x])^(-1),x]
```

output
```
x/5 - (2*((-3*Log[3 + Tan[(c + d*x)/2]])/8 + (3*Log[1 + 3*Tan[(c + d*x)/2]
])/8))/(5*d)
```

3.53.3.1 Defintions of rubi rules used

rule 1081
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> With[{q = Rt[b^2
- 4*a*c, 2]}, Simp[c   Int[ExpandIntegrand[1/((b/2 - q/2 + c*x)*(b/2 + q/2
+ c*x)), x], x], x]] /; FreeQ[{a, b, c}, x] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3139
```
Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{e = Fre
eFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d)   Subst[Int[1/(a + 2*b*e*x + a
*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[a^2 - b^2, 0]
```

rule 4270
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(-1), x_Symbol] :> Simp[x/a, x]
- Simp[1/a   Int[1/(1 + (a/b)*Sin[c + d*x]), x], x] /; FreeQ[{a, b, c, d},
x] && NeQ[a^2 - b^2, 0]
```

3.53.4 Maple [A] (verified)

Time = 0.31 (sec) , antiderivative size = 41, normalized size of antiderivative = 0.60

method	result	size
norman	$\frac{x}{5}+\frac{3\ln\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+3\right)}{20d}-\frac{3\ln\left(3\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)}{20d}$	41
risch	$\frac{x}{5}+\frac{3\ln\left(\frac{4}{5}+\frac{3i}{5}+e^{i(dx+c)}\right)}{20d}-\frac{3\ln\left(e^{i(dx+c)}-\frac{4}{5}+\frac{3i}{5}\right)}{20d}$	43
parallelrisch	$\frac{3\ln\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+3\right)-3\ln\left(3\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)+\ln(27)+4dx}{20d}$	43
derivativedivides	$\frac{-\frac{3\ln\left(3\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)}{20}+\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{5}+\frac{3\ln\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+3\right)}{20}}{d}$	48
default	$\frac{-\frac{3\ln\left(3\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+1\right)}{20}+\frac{2\arctan\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)\right)}{5}+\frac{3\ln\left(\tan\left(\frac{dx}{2}+\frac{c}{2}\right)+3\right)}{20}}{d}$	48

input
```
int(1/(5+3*csc(d*x+c)),x,method=_RETURNVERBOSE)
```

output
```
1/5*x+3/20/d*ln(tan(1/2*d*x+1/2*c)+3)-3/20/d*ln(3*tan(1/2*d*x+1/2*c)+1)
```

3.53.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.76

$$\int \frac{1}{5+3\csc(c+dx)}\,dx = \frac{8\,dx + 3\,\log\left(4\,\cos\left(dx+c\right) + 3\,\sin\left(dx+c\right) + 5\right) - 3\,\log\left(-4\,\cos\left(dx+c\right) + 3\,\sin\left(dx+c\right) + 5\right)}{40\,d}$$

input
```
integrate(1/(5+3*csc(d*x+c)),x, algorithm="fricas")
```

output
```
1/40*(8*d*x + 3*log(4*cos(d*x + c) + 3*sin(d*x + c) + 5) - 3*log(-4*cos(d*
x + c) + 3*sin(d*x + c) + 5))/d
```

3.53.6 Sympy [F]

$$\int \frac{1}{5+3\csc(c+dx)}\,dx = \int \frac{1}{3\csc\left(c+dx\right)+5}\,dx$$

input
```
integrate(1/(5+3*csc(d*x+c)),x)
```

output
```
Integral(1/(3*csc(c + d*x) + 5), x)
```

3.53.7 Maxima [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 71, normalized size of antiderivative = 1.04

$$\int \frac{1}{5+3\csc(c+dx)}\,dx = \frac{8\,\arctan\left(\frac{\sin(dx+c)}{\cos(dx+c)+1}\right) - 3\,\log\left(\frac{3\,\sin(dx+c)}{\cos(dx+c)+1}+1\right) + 3\,\log\left(\frac{\sin(dx+c)}{\cos(dx+c)+1}+3\right)}{20\,d}$$

input
```
integrate(1/(5+3*csc(d*x+c)),x, algorithm="maxima")
```

output
```
1/20*(8*arctan(sin(d*x + c)/(cos(d*x + c) + 1)) - 3*log(3*sin(d*x + c)/(co
s(d*x + c) + 1) + 1) + 3*log(sin(d*x + c)/(cos(d*x + c) + 1) + 3))/d
```

3.53.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.66

$$\int \frac{1}{5+3\csc(c+dx)}\,dx = \frac{4\,dx+4\,c-3\,\log\left(\left|3\,\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+1\right|\right)+3\,\log\left(\left|\tan\left(\frac{1}{2}\,dx+\frac{1}{2}\,c\right)+3\right|\right)}{20\,d}$$

input
```
integrate(1/(5+3*csc(d*x+c)),x, algorithm="giac")
```

output
```
1/20*(4*d*x + 4*c - 3*log(abs(3*tan(1/2*d*x + 1/2*c) + 1)) + 3*log(abs(tan
(1/2*d*x + 1/2*c) + 3)))/d
```

3.53.9 Mupad [B] (verification not implemented)

Time = 18.57 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.40

$$\int \frac{1}{5+3\csc(c+dx)}\,dx = \frac{x}{5}-\frac{3\,\mathrm{atanh}\left(\frac{1}{2\left(\frac{200\tan\left(\frac{c}{2}+\frac{d\,x}{2}\right)}{27}+\frac{20}{9}\right)}+\frac{41}{40}\right)}{10\,d}$$

input
```
int(1/(3/sin(c + d*x) + 5),x)
```

output
```
x/5 - (3*atanh(1/(2*((200*tan(c/2 + (d*x)/2))/27 + 20/9)) + 41/40))/(10*d)
```

3.54 $\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx$

3.54.1 Optimal result

Integrand size = 21, antiderivative size = 274

$$\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx = -\frac{\cot(e+fx)(a+b\csc(e+fx))^{1+m}}{bf(2+m)}$$
$$+\frac{\sqrt{2}a(a+b)\operatorname{AppellF1}\left(\frac{1}{2},\frac{1}{2},-1-m,\frac{3}{2},\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)\cot(e+fx)(a+b\csc(e+fx}{b^2f(2+m)\sqrt{1+\csc(e+fx)}}$$
$$-\frac{\sqrt{2}(a^2+b^2(1+m))\operatorname{AppellF1}\left(\frac{1}{2},\frac{1}{2},-m,\frac{3}{2},\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)\cot(e+fx)(a+b\csc(e}{b^2f(2+m)\sqrt{1+\csc(e+fx)}}$$

output
```
-cot(f*x+e)*(a+b*csc(f*x+e))^(1+m)/b/f/(2+m)+a*(a+b)*AppellF1(1/2,-1-m,1/2
,3/2,b*(1-csc(f*x+e))/(a+b),1/2-1/2*csc(f*x+e))*cot(f*x+e)*(a+b*csc(f*x+e)
)^m*2^(1/2)/b^2/f/(2+m)/(((a+b*csc(f*x+e))/(a+b))^m)/(1+csc(f*x+e))^(1/2)-
(a^2+b^2*(1+m))*AppellF1(1/2,-m,1/2,3/2,b*(1-csc(f*x+e))/(a+b),1/2-1/2*csc
(f*x+e))*cot(f*x+e)*(a+b*csc(f*x+e))^m*2^(1/2)/b^2/f/(2+m)/(((a+b*csc(f*x+
e))/(a+b))^m)/(1+csc(f*x+e))^(1/2)
```

3.54.2 Mathematica [F]

$$\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx=\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx$$

input
```
Integrate[Csc[e + f*x]^3*(a + b*Csc[e + f*x])^m,x]
```

output
```
Integrate[Csc[e + f*x]^3*(a + b*Csc[e + f*x])^m, x]
```

3.54.3 Rubi [A] (verified)

Time = 0.69 (sec) , antiderivative size = 274, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.381, Rules used = {3042, 4327, 3042, 4495, 3042, 4321, 156, 155}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \csc(e+fx)^3(a+b\csc(e+fx))^m dx$$

$$\Big\downarrow \text{4327}$$

$$\frac{\int \csc(e+fx)(b(m+1)-a\csc(e+fx))(a+b\csc(e+fx))^m dx}{b(m+2)}-\frac{\cot(e+fx)(a+b\csc(e+fx))^{m+1}}{bf(m+2)}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \csc(e+fx)(b(m+1)-a\csc(e+fx))(a+b\csc(e+fx))^m dx}{b(m+2)}-\frac{\cot(e+fx)(a+b\csc(e+fx))^{m+1}}{bf(m+2)}$$

$$\Big\downarrow \text{4495}$$

$$\frac{\frac{\left(a^2+b^2(m+1)\right)\int \csc(e+fx)(a+b\csc(e+fx))^m dx}{b}-\frac{a\int \csc(e+fx)(a+b\csc(e+fx))^{m+1}dx}{b}}{b(m+2)}-\frac{\cot(e+fx)(a+b\csc(e+fx))^{m+1}}{bf(m+2)}$$

$\Big\downarrow$ 3042

$$\frac{\frac{\left(a^2+b^2(m+1)\right)\int \csc(e+fx)(a+b\csc(e+fx))^m dx}{b}-\frac{a\int \csc(e+fx)(a+b\csc(e+fx))^{m+1}dx}{b}}{b(m+2)}-\frac{\cot(e+fx)(a+b\csc(e+fx))^{m+1}}{bf(m+2)}$$

$\Big\downarrow$ 4321

$$\frac{\frac{\left(a^2+b^2(m+1)\right)\cot(e+fx)\int \frac{(a+b\csc(e+fx))^m}{\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}d\csc(e+fx)}{bf\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}-\frac{a\cot(e+fx)\int \frac{(a+b\csc(e+fx))^{m+1}}{\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}d\csc(e+fx)}{bf\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}}{b(m+2)}-\frac{\cot(e+fx)(a+b\csc(e+fx))^{m+1}}{bf(m+2)}$$

$\Big\downarrow$ 156

$$\frac{\frac{\left(a^2+b^2(m+1)\right)\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}\int \frac{\left(\frac{a}{a+b}+\frac{b\csc(e+fx)}{a+b}\right)^m}{\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}d\csc(e+fx)}{bf\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}-\frac{a(a+b)\cot(e+fx)(a+b\csc(e+fx)}{b}}{b(m+2)}-\frac{\cot(e+fx)(a+b\csc(e+fx))^{m+1}}{bf(m+2)}$$

$\Big\downarrow$ 155

$$\frac{\frac{\sqrt{2}a(a+b)\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}\text{AppellF1}\left(\frac{1}{2},\frac{1}{2},-m-1,\frac{3}{2},\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)}{bf\sqrt{\csc(e+fx)+1}}-\frac{\sqrt{2}\left(a^2+b^2(m+1)\right)\text{co}}{}}{b(m+2)}-\frac{\cot(e+fx)(a+b\csc(e+fx))^{m+1}}{bf(m+2)}$$

input
```
Int[Csc[e + f*x]^3*(a + b*Csc[e + f*x])^m,x]
```

3.54. $\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx$

output
```
-((Cot[e + f*x]*(a + b*Csc[e + f*x])^(1 + m))/(b*f*(2 + m))) + ((Sqrt[2]*a
*(a + b)*AppellF1[1/2, 1/2, -1 - m, 3/2, (1 - Csc[e + f*x])/2, (b*(1 - Csc
[e + f*x]))/(a + b)]*Cot[e + f*x]*(a + b*Csc[e + f*x])^m)/(b*f*Sqrt[1 + Cs
c[e + f*x]]*((a + b*Csc[e + f*x])/(a + b))^m) - (Sqrt[2]*(a^2 + b^2*(1 + m
))*AppellF1[1/2, 1/2, -m, 3/2, (1 - Csc[e + f*x])/2, (b*(1 - Csc[e + f*x])
)/(a + b)]*Cot[e + f*x]*(a + b*Csc[e + f*x])^m)/(b*f*Sqrt[1 + Csc[e + f*x]
]*((a + b*Csc[e + f*x])/(a + b))^m))/(b*(2 + m))
```

3.54.3.1 Defintions of rubi rules used

rule 155
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[((a + b*x)^(m + 1)/(b*(m + 1)*Simplify[b/(b*c - a*d)]^n*
Simplify[b/(b*e - a*f)]^p))*AppellF1[m + 1, -n, -p, m + 2, (-d)*((a + b*x)/
(b*c - a*d)), (-f)*((a + b*x)/(b*e - a*f))], x] /; FreeQ[{a, b, c, d, e, f,
 m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] && GtQ[Sim
plify[b/(b*c - a*d)], 0] && GtQ[Simplify[b/(b*e - a*f)], 0] && !(GtQ[Simpl
ify[d/(d*a - c*b)], 0] && GtQ[Simplify[d/(d*e - c*f)], 0] && SimplerQ[c + d
*x, a + b*x]) && !(GtQ[Simplify[f/(f*a - e*b)], 0] && GtQ[Simplify[f/(f*c
- e*d)], 0] && SimplerQ[e + f*x, a + b*x])
```

rule 156
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[(e + f*x)^FracPart[p]/(Simplify[b/(b*e - a*f)]^IntPart[p
]*(b*((e + f*x)/(b*e - a*f)))^FracPart[p])   Int[(a + b*x)^m*(c + d*x)^n*Si
mp[b*(e/(b*e - a*f)) + b*f*(x/(b*e - a*f)), x]^p, x], x] /; FreeQ[{a, b, c,
 d, e, f, m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] &
& GtQ[Simplify[b/(b*c - a*d)], 0] && !GtQ[Simplify[b/(b*e - a*f)], 0]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4321
```
Int[csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_
Symbol] :> Simp[Cot[e + f*x]/(f*Sqrt[1 + Csc[e + f*x]]*Sqrt[1 - Csc[e + f*x
]])   Subst[Int[(a + b*x)^m/(Sqrt[1 + x]*Sqrt[1 - x]), x], x, Csc[e + f*x]]
, x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^2, 0] && !IntegerQ[2*m]
```

rule 4327
```
Int[csc[(e_.) + (f_.)*(x_)]^3*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_),
x_Symbol] :> Simp[(-Cot[e + f*x])*((a + b*Csc[e + f*x])^(m + 1)/(b*f*(m + 2
))), x] + Simp[1/(b*(m + 2))   Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m*(b*(
m + 1) - a*Csc[e + f*x]), x], x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 -
b^2, 0] &&  !LtQ[m, -1]
```

rule 4495
```
Int[csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_)*(cs
c[(e_.) + (f_.)*(x_)]*(B_.) + (A_)), x_Symbol] :> Simp[(A*b - a*B)/b   Int[
Csc[e + f*x]*(a + b*Csc[e + f*x])^m, x], x] + Simp[B/b   Int[Csc[e + f*x]*(
a + b*Csc[e + f*x])^(m + 1), x], x] /; FreeQ[{a, b, A, B, e, f, m}, x] && N
eQ[A*b - a*B, 0] && NeQ[a^2 - b^2, 0]
```

3.54.4 Maple [F]

$$\int \csc\left(fx+e\right)^{3} \left(a+b\csc\left(fx+e\right)\right)^{m} dx$$

input
```
int(csc(f*x+e)^3*(a+b*csc(f*x+e))^m,x)
```

output
```
int(csc(f*x+e)^3*(a+b*csc(f*x+e))^m,x)
```

3.54.5 Fricas [F]

$$\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx=\int \left(b\csc\left(fx+e\right)+a\right)^{m} \csc\left(fx+e\right)^{3}\,dx$$

input
```
integrate(csc(f*x+e)^3*(a+b*csc(f*x+e))^m,x, algorithm="fricas")
```

output
```
integral((b*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

3.54.6 Sympy [F]

$$\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx=\int (a+b\csc{(e+fx)})^m\csc^3{(e+fx)}\,dx$$

input
```
integrate(csc(f*x+e)**3*(a+b*csc(f*x+e))**m,x)
```

output
```
Integral((a + b*csc(e + f*x))**m*csc(e + f*x)**3, x)
```

3.54.7 Maxima [F]

$$\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx=\int (b\csc{(fx+e)}+a)^m\csc{(fx+e)}^3\,dx$$

input
```
integrate(csc(f*x+e)^3*(a+b*csc(f*x+e))^m,x, algorithm="maxima")
```

output
```
integrate((b*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

3.54.8 Giac [F]

$$\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx=\int (b\csc{(fx+e)}+a)^m\csc{(fx+e)}^3\,dx$$

input
```
integrate(csc(f*x+e)^3*(a+b*csc(f*x+e))^m,x, algorithm="giac")
```

output
```
integrate((b*csc(f*x + e) + a)^m*csc(f*x + e)^3, x)
```

3.54.9 Mupad [F(-1)]

Timed out.

$$\int \csc^3(e+fx)(a+b\csc(e+fx))^m\,dx=\int \frac{\left(a+\frac{b}{\sin(e+f\,x)}\right)^m}{\sin\left(e+f\,x\right)^3}\,dx$$

input
```
int((a + b/sin(e + f*x))^m/sin(e + f*x)^3,x)
```

output
```
int((a + b/sin(e + f*x))^m/sin(e + f*x)^3, x)
```

3.55 $\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx$

3.55.1 Optimal result

Integrand size = 21, antiderivative size = 220

$$\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx = -\frac{\sqrt{2}(a+b)\operatorname{AppellF1}\left(\frac{1}{2},\frac{1}{2},-1-m,\frac{3}{2},\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)\cot(e+fx)(a+b\csc(e+fx)}{bf\sqrt{1+\csc(e+fx)}} + \frac{\sqrt{2}a\operatorname{AppellF1}\left(\frac{1}{2},\frac{1}{2},-m,\frac{3}{2},\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(}{a+}\right.}{bf\sqrt{1+\csc(e+fx)}}$$

output
```
-(a+b)*AppellF1(1/2,-1-m,1/2,3/2,b*(1-csc(f*x+e))/(a+b),1/2-1/2*csc(f*x+e)
)*cot(f*x+e)*(a+b*csc(f*x+e))^m*2^(1/2)/b/f/(((a+b*csc(f*x+e))/(a+b))^m)/(
1+csc(f*x+e))^(1/2)+a*AppellF1(1/2,-m,1/2,3/2,b*(1-csc(f*x+e))/(a+b),1/2-1
/2*csc(f*x+e))*cot(f*x+e)*(a+b*csc(f*x+e))^m*2^(1/2)/b/f/(((a+b*csc(f*x+e)
)/(a+b))^m)/(1+csc(f*x+e))^(1/2)
```

3.55.2 Mathematica [F]

$$\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx=\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx$$

input
```
Integrate[Csc[e + f*x]^2*(a + b*Csc[e + f*x])^m,x]
```

output
```
Integrate[Csc[e + f*x]^2*(a + b*Csc[e + f*x])^m, x]
```

3.55.3 Rubi [A] (verified)

Time = 0.45 (sec) , antiderivative size = 220, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.286, Rules used = {3042, 4325, 3042, 4321, 156, 155}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \csc(e+fx)^2(a+b\csc(e+fx))^m dx$$

$$\Big\downarrow \text{4325}$$

$$\frac{\int \csc(e+fx)(a+b\csc(e+fx))^{m+1}dx}{b}-\frac{a\int \csc(e+fx)(a+b\csc(e+fx))^m dx}{b}$$

$$\Big\downarrow \text{3042}$$

$$\frac{\int \csc(e+fx)(a+b\csc(e+fx))^{m+1}dx}{b}-\frac{a\int \csc(e+fx)(a+b\csc(e+fx))^m dx}{b}$$

$$\Big\downarrow \text{4321}$$

$$\frac{\cot(e+fx)\int \frac{(a+b\csc(e+fx))^{m+1}}{\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}d\csc(e+fx)}{bf\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}-\frac{a\cot(e+fx)\int \frac{(a+b\csc(e+fx))^m}{\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}d\csc(e+fx)}{bf\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}$$

$$\Big\downarrow \text{156}$$

$$\frac{(a+b)\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}\int\frac{\left(\frac{a}{a+b}+\frac{b\csc(e+fx)}{a+b}\right)^{m+1}}{\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}d\csc(e+fx)}{bf\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}-\frac{a\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}\int\frac{\left(\frac{a}{a+b}+\frac{b\csc(e+fx)}{a+b}\right)^{m}}{\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}d\csc(e+fx)}{bf\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}$$

$$\Big\downarrow \text{155}$$

$$\frac{\sqrt{2}a\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}\text{AppellF1}\left(\frac{1}{2},\frac{1}{2},-m,\frac{3}{2},\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right.}{bf\sqrt{\csc(e+fx)+1}}$$

$$\frac{\sqrt{2}(a+b)\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}\text{AppellF1}\left(\frac{1}{2},\frac{1}{2},-m-1,\frac{3}{2},\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-}{}\right.}{bf\sqrt{\csc(e+fx)+1}}$$

input
```
Int[Csc[e + f*x]^2*(a + b*Csc[e + f*x])^m,x]
```

output
```
-((Sqrt[2]*(a + b)*AppellF1[1/2, 1/2, -1 - m, 3/2, (1 - Csc[e + f*x])/2, (
b*(1 - Csc[e + f*x]))/(a + b)]*Cot[e + f*x]*(a + b*Csc[e + f*x])^m)/(b*f*S
qrt[1 + Csc[e + f*x]]*((a + b*Csc[e + f*x])/(a + b))^m)) + (Sqrt[2]*a*Appe
llF1[1/2, 1/2, -m, 3/2, (1 - Csc[e + f*x])/2, (b*(1 - Csc[e + f*x]))/(a +
b)]*Cot[e + f*x]*(a + b*Csc[e + f*x])^m)/(b*f*Sqrt[1 + Csc[e + f*x]]*((a +
 b*Csc[e + f*x])/(a + b))^m)
```

3.55.3.1 Defintions of rubi rules used

rule 155
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[((a + b*x)^(m + 1)/(b*(m + 1)*Simplify[b/(b*c - a*d)]^n*
Simplify[b/(b*e - a*f)]^p))*AppellF1[m + 1, -n, -p, m + 2, (-d)*((a + b*x)/
(b*c - a*d)), (-f)*((a + b*x)/(b*e - a*f))], x] /; FreeQ[{a, b, c, d, e, f,
 m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] && GtQ[Sim
plify[b/(b*c - a*d)], 0] && GtQ[Simplify[b/(b*e - a*f)], 0] && !(GtQ[Simpl
ify[d/(d*a - c*b)], 0] && GtQ[Simplify[d/(d*e - c*f)], 0] && SimplerQ[c + d
*x, a + b*x]) && !(GtQ[Simplify[f/(f*a - e*b)], 0] && GtQ[Simplify[f/(f*c
- e*d)], 0] && SimplerQ[e + f*x, a + b*x])
```

rule 156
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[(e + f*x)^FracPart[p]/(Simplify[b/(b*e - a*f)]^IntPart[p
]*(b*((e + f*x)/(b*e - a*f)))^FracPart[p])   Int[(a + b*x)^m*(c + d*x)^n*Si
mp[b*(e/(b*e - a*f)) + b*f*(x/(b*e - a*f)), x]^p, x], x] /; FreeQ[{a, b, c,
 d, e, f, m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] &
& GtQ[Simplify[b/(b*c - a*d)], 0] && !GtQ[Simplify[b/(b*e - a*f)], 0]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4321
```
Int[csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_
Symbol] :> Simp[Cot[e + f*x]/(f*Sqrt[1 + Csc[e + f*x]]*Sqrt[1 - Csc[e + f*x
]])   Subst[Int[(a + b*x)^m/(Sqrt[1 + x]*Sqrt[1 - x]), x], x, Csc[e + f*x]]
, x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^2, 0] && !IntegerQ[2*m]
```

rule 4325
```
Int[csc[(e_.) + (f_.)*(x_)]^2*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_),
x_Symbol] :> Simp[-a/b   Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m, x], x] +
Simp[1/b   Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^(m + 1), x], x] /; FreeQ[{
a, b, e, f, m}, x] && NeQ[a^2 - b^2, 0]
```

3.55.4 Maple [F]

$$\int \csc (fx + e)^2 (a + b\csc (fx + e))^m \, dx$$

input
```
int(csc(f*x+e)^2*(a+b*csc(f*x+e))^m,x)
```

output
```
int(csc(f*x+e)^2*(a+b*csc(f*x+e))^m,x)
```

3.55.5 Fricas [F]

$$\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx = \int (b\csc(fx+e)+a)^m \csc(fx+e)^2\,dx$$

input
```
integrate(csc(f*x+e)^2*(a+b*csc(f*x+e))^m,x, algorithm="fricas")
```

output
```
integral((b*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

3.55.6 Sympy [F]

$$\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx = \int (a+b\csc(e+fx))^m \csc^2(e+fx)\,dx$$

input
```
integrate(csc(f*x+e)**2*(a+b*csc(f*x+e))**m,x)
```

output
```
Integral((a + b*csc(e + f*x))**m*csc(e + f*x)**2, x)
```

3.55.7 Maxima [F]

$$\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx = \int (b\csc(fx+e)+a)^m \csc(fx+e)^2\,dx$$

input
```
integrate(csc(f*x+e)^2*(a+b*csc(f*x+e))^m,x, algorithm="maxima")
```

output
```
integrate((b*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

3.55.8 Giac [F]

$$\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx=\int (b\csc(fx+e)+a)^m\csc(fx+e)^2\,dx$$

input
```
integrate(csc(f*x+e)^2*(a+b*csc(f*x+e))^m,x, algorithm="giac")
```

output
```
integrate((b*csc(f*x + e) + a)^m*csc(f*x + e)^2, x)
```

3.55.9 Mupad [F(-1)]

Timed out.

$$\int \csc^2(e+fx)(a+b\csc(e+fx))^m\,dx=\int \frac{\left(a+\frac{b}{\sin(e+f\,x)}\right)^m}{\sin(e+f\,x)^2}\,dx$$

input
```
int((a + b/sin(e + f*x))^m/sin(e + f*x)^2,x)
```

output
```
int((a + b/sin(e + f*x))^m/sin(e + f*x)^2, x)
```

3.56 $\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx$

3.56.1 Optimal result

Integrand size = 19, antiderivative size = 104

$$\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx =$$

$$-\frac{\sqrt{2}\,\text{AppellF1}\left(\frac{1}{2},\frac{1}{2},-m,\frac{3}{2},\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right)\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e}{a+b}\right.}{f\sqrt{1+\csc(e+fx)}}$$

output
```
-AppellF1(1/2,-m,1/2,3/2,b*(1-csc(f*x+e))/(a+b),1/2-1/2*csc(f*x+e))*cot(f*
x+e)*(a+b*csc(f*x+e))^m*2^(1/2)/f/(((a+b*csc(f*x+e))/(a+b))^m)/(1+csc(f*x+
e))^(1/2)
```

3.56.2 Mathematica [F]

$$\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx = \int \csc(e+fx)(a+b\csc(e+fx))^m\,dx$$

input
```
Integrate[Csc[e + f*x]*(a + b*Csc[e + f*x])^m,x]
```

output
```
Integrate[Csc[e + f*x]*(a + b*Csc[e + f*x])^m, x]
```

3.56.3 Rubi [A] (verified)

Time = 0.25 (sec) , antiderivative size = 104, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.211$, Rules used = {3042, 4321, 156, 155}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \csc(e+fx)(a+b\csc(e+fx))^m dx$$

$$\Big\downarrow \text{4321}$$

$$\frac{\cot(e+fx)\int \frac{(a+b\csc(e+fx))^m}{\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}d\csc(e+fx)}{f\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}$$

$$\Big\downarrow \text{156}$$

$$\frac{\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}\int \frac{\left(\frac{a}{a+b}+\frac{b\csc(e+fx)}{a+b}\right)^m}{\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}d\csc(e+fx)}{f\sqrt{1-\csc(e+fx)}\sqrt{\csc(e+fx)+1}}$$

$$\Big\downarrow \text{155}$$

$$-\frac{\sqrt{2}\cot(e+fx)(a+b\csc(e+fx))^m\left(\frac{a+b\csc(e+fx)}{a+b}\right)^{-m}\text{AppellF1}\left(\frac{1}{2},\frac{1}{2},-m,\frac{3}{2},\frac{1}{2}(1-\csc(e+fx)),\frac{b(1-\csc(e+fx))}{a+b}\right.}{f\sqrt{\csc(e+fx)+1}}$$

input
```
Int[Csc[e + f*x]*(a + b*Csc[e + f*x])^m,x]
```

output
```
-((Sqrt[2]*AppellF1[1/2, 1/2, -m, 3/2, (1 - Csc[e + f*x])/2, (b*(1 - Csc[e
 + f*x]))/(a + b)]*Cot[e + f*x]*(a + b*Csc[e + f*x])^m)/(f*Sqrt[1 + Csc[e
+ f*x]]*((a + b*Csc[e + f*x])/(a + b))^m))
```

3.56.3.1 Defintions of rubi rules used

rule 155
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[((a + b*x)^(m + 1)/(b*(m + 1)*Simplify[b/(b*c - a*d)]^n*
Simplify[b/(b*e - a*f)]^p))*AppellF1[m + 1, -n, -p, m + 2, (-d)*((a + b*x)/
(b*c - a*d)), (-f)*((a + b*x)/(b*e - a*f))], x] /; FreeQ[{a, b, c, d, e, f,
 m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] && GtQ[Sim
plify[b/(b*c - a*d)], 0] && GtQ[Simplify[b/(b*e - a*f)], 0] && !(GtQ[Simpl
ify[d/(d*a - c*b)], 0] && GtQ[Simplify[d/(d*e - c*f)], 0] && SimplerQ[c + d
*x, a + b*x]) && !(GtQ[Simplify[f/(f*a - e*b)], 0] && GtQ[Simplify[f/(f*c
- e*d)], 0] && SimplerQ[e + f*x, a + b*x])
```

rule 156
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))
^(p_), x_] :> Simp[(e + f*x)^FracPart[p]/(Simplify[b/(b*e - a*f)]^IntPart[p
]*(b*((e + f*x)/(b*e - a*f)))^FracPart[p])   Int[(a + b*x)^m*(c + d*x)^n*Si
mp[b*(e/(b*e - a*f)) + b*f*(x/(b*e - a*f)), x]^p, x], x] /; FreeQ[{a, b, c,
 d, e, f, m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] &
& GtQ[Simplify[b/(b*c - a*d)], 0] && !GtQ[Simplify[b/(b*e - a*f)], 0]
```

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4321
```
Int[csc[(e_.) + (f_.)*(x_)]*(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_))^(m_), x_
Symbol] :> Simp[Cot[e + f*x]/(f*Sqrt[1 + Csc[e + f*x]]*Sqrt[1 - Csc[e + f*x
]])   Subst[Int[(a + b*x)^m/(Sqrt[1 + x]*Sqrt[1 - x]), x], x, Csc[e + f*x]]
, x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^2, 0] && !IntegerQ[2*m]
```

3.56.4 Maple [F]

$$\int \csc(fx+e)\,(a+b\csc(fx+e))^m\,dx$$

input
```
int(csc(f*x+e)*(a+b*csc(f*x+e))^m,x)
```

output
```
int(csc(f*x+e)*(a+b*csc(f*x+e))^m,x)
```

3.56.5 Fricas [F]

$$\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx = \int (b\csc(fx+e)+a)^m \csc(fx+e)\,dx$$

input
```
integrate(csc(f*x+e)*(a+b*csc(f*x+e))^m,x, algorithm="fricas")
```

output
```
integral((b*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

3.56.6 Sympy [F]

$$\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx = \int (a+b\csc(e+fx))^m \csc(e+fx)\,dx$$

input
```
integrate(csc(f*x+e)*(a+b*csc(f*x+e))**m,x)
```

output
```
Integral((a + b*csc(e + f*x))**m*csc(e + f*x), x)
```

3.56.7 Maxima [F]

$$\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx = \int (b\csc(fx+e)+a)^m \csc(fx+e)\,dx$$

input
```
integrate(csc(f*x+e)*(a+b*csc(f*x+e))^m,x, algorithm="maxima")
```

output
```
integrate((b*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

3.56.8 Giac [F]

$$\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx = \int (b\csc(fx+e)+a)^m\csc(fx+e)\,dx$$

input
```
integrate(csc(f*x+e)*(a+b*csc(f*x+e))^m,x, algorithm="giac")
```

output
```
integrate((b*csc(f*x + e) + a)^m*csc(f*x + e), x)
```

3.56.9 Mupad [F(-1)]

Timed out.

$$\int \csc(e+fx)(a+b\csc(e+fx))^m\,dx = \int \frac{\left(a+\frac{b}{\sin(e+f\,x)}\right)^m}{\sin(e+f\,x)}\,dx$$

input
```
int((a + b/sin(e + f*x))^m/sin(e + f*x),x)
```

output
```
int((a + b/sin(e + f*x))^m/sin(e + f*x), x)
```

3.57 $\int (a+b\csc(e+fx))^m\,dx$

3.57.1 Optimal result

Integrand size = 12, antiderivative size = 12

$$\int (a+b\csc(e+fx))^m\,dx = \text{Int}((a+b\csc(e+fx))^m,x)$$

output
```
Unintegrable((a+b*csc(f*x+e))^m,x)
```

3.57.2 Mathematica [N/A]

Not integrable

Time = 3.16 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.17

$$\int (a+b\csc(e+fx))^m\,dx = \int (a+b\csc(e+fx))^m\,dx$$

input
```
Integrate[(a + b*Csc[e + f*x])^m,x]
```

output
```
Integrate[(a + b*Csc[e + f*x])^m, x]
```

3.57.3 Rubi [N/A]

Not integrable

Time = 0.17 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 0, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.000, Rules used = {3042, 4273}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (a+b\csc(e+fx))^m\,dx$$

$$\Big\downarrow \; 3042$$

$$\int (a+b\csc(e+fx))^m dx$$

$$\Big\downarrow \; 4273$$

$$\int (a+b\csc(e+fx))^m dx$$

input
```
Int[(a + b*Csc[e + f*x])^m,x]
```

output
```
$Aborted
```

3.57.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4273
```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_))^(n_), x_Symbol] :> Unintegrable[
(a + b*Csc[c + d*x])^n, x] /; FreeQ[{a, b, c, d, n}, x] && NeQ[a^2 - b^2, 0
] && !IntegerQ[2*n]
```

3.57.4 Maple [N/A] (verified)

Not integrable

Time = 0.61 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.00

$$\int (a + b\csc (fx + e))^m \, dx$$

input
```
int((a+b*csc(f*x+e))^m,x)
```

output
```
int((a+b*csc(f*x+e))^m,x)
```

3.57.5 Fricas [N/A]

Not integrable

Time = 0.26 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.17

$$\int (a + b\csc(e + fx))^m \, dx = \int (b\csc (fx + e) + a)^m \, dx$$

input
```
integrate((a+b*csc(f*x+e))^m,x, algorithm="fricas")
```

output
```
integral((b*csc(f*x + e) + a)^m, x)
```

3.57.6 Sympy [N/A]

Not integrable

Time = 0.50 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.00

$$\int (a + b\csc(e + fx))^m \, dx = \int (a + b\csc (e + fx))^m \, dx$$

input
```
integrate((a+b*csc(f*x+e))**m,x)
```

output
```
Integral((a + b*csc(e + f*x))**m, x)
```

3.57.7 Maxima [N/A]

Not integrable

Time = 1.24 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.17

$$\int (a+b\csc(e+fx))^m\,dx = \int (b\csc(fx+e)+a)^m\,dx$$

input
```
integrate((a+b*csc(f*x+e))^m,x, algorithm="maxima")
```

output
```
integrate((b*csc(f*x + e) + a)^m, x)
```

3.57.8 Giac [N/A]

Not integrable

Time = 0.32 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.17

$$\int (a+b\csc(e+fx))^m\,dx = \int (b\csc(fx+e)+a)^m\,dx$$

input
```
integrate((a+b*csc(f*x+e))^m,x, algorithm="giac")
```

output
```
integrate((b*csc(f*x + e) + a)^m, x)
```

3.57.9 Mupad [N/A]

Not integrable

Time = 18.73 (sec) , antiderivative size = 16, normalized size of antiderivative = 1.33

$$\int (a+b\csc(e+fx))^m\,dx = \int \left(a+\frac{b}{\sin(e+f\,x)}\right)^m\,dx$$

input
```
int((a + b/sin(e + f*x))^m,x)
```

output
```
int((a + b/sin(e + f*x))^m, x)
```

3.58 $\int (a+b\csc(e+fx))^m \sin(e+fx)\,dx$

3.58.1 Optimal result

Integrand size = 19, antiderivative size = 19

$$\int (a+b\csc(e+fx))^m \sin(e+fx)\,dx = \text{Int}((a+b\csc(e+fx))^m \sin(e+fx),x)$$

output
```
Unintegrable((a+b*csc(f*x+e))^m*sin(f*x+e),x)
```

3.58.2 Mathematica [N/A]

Not integrable

Time = 8.30 (sec) , antiderivative size = 21, normalized size of antiderivative = 1.11

$$\int (a+b\csc(e+fx))^m \sin(e+fx)\,dx = \int (a+b\csc(e+fx))^m \sin(e+fx)\,dx$$

input
```
Integrate[(a + b*Csc[e + f*x])^m*Sin[e + f*x],x]
```

output
```
Integrate[(a + b*Csc[e + f*x])^m*Sin[e + f*x], x]
```

3.58.3 Rubi [N/A]

Not integrable

Time = 0.19 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 0, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.000, Rules used = {3042, 4357}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sin(e+fx)(a+b\csc(e+fx))^m\,dx$$

$$\Big\downarrow \text{3042}$$

$$\int \frac{(a+b\csc(e+fx))^m}{\csc(e+fx)}dx$$

$$\Big\downarrow \text{4357}$$

$$\int \sin(e+fx)(a+b\csc(e+fx))^m dx$$

input
```
Int[(a + b*Csc[e + f*x])^m*Sin[e + f*x],x]
```

output
```
$Aborted
```

3.58.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4357
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_.), x_Symbol] :> Unintegrable[(d*Csc[e + f*x])^n*(a + b*Csc[e + f*
x])^m, x] /; FreeQ[{a, b, d, e, f, m, n}, x]
```

3.58.4 Maple [N/A] (verified)

Not integrable

Time = 0.68 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.00

$$\int (a+b\csc (fx+e))^{m} \sin (fx+e) \, dx$$

input
```
int((a+b*csc(f*x+e))^m*sin(f*x+e),x)
```

output
```
int((a+b*csc(f*x+e))^m*sin(f*x+e),x)
```

3.58.5 Fricas [N/A]

Not integrable

Time = 0.26 (sec) , antiderivative size = 21, normalized size of antiderivative = 1.11

$$\int (a+b\csc(e+fx))^m\sin(e+fx)\,dx = \int (b\csc (fx+e)+a)^{m} \sin (fx+e) \, dx$$

input
```
integrate((a+b*csc(f*x+e))^m*sin(f*x+e),x, algorithm="fricas")
```

output
```
integral((b*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

3.58.6 Sympy [N/A]

Not integrable

Time = 4.48 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.00

$$\int (a+b\csc(e+fx))^m\sin(e+fx)\,dx = \int (a+b\csc (e+fx))^{m} \sin (e+fx) \, dx$$

input
```
integrate((a+b*csc(f*x+e))**m*sin(f*x+e),x)
```

output
```
Integral((a + b*csc(e + f*x))**m*sin(e + f*x), x)
```

3.58.7 Maxima [N/A]

Not integrable

Time = 1.65 (sec) , antiderivative size = 21, normalized size of antiderivative = 1.11

$$\int (a+b\csc(e+fx))^m\sin(e+fx)\,dx = \int (b\csc(fx+e)+a)^m\sin(fx+e)\,dx$$

input
```
integrate((a+b*csc(f*x+e))^m*sin(f*x+e),x, algorithm="maxima")
```

output
```
integrate((b*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

3.58.8 Giac [N/A]

Not integrable

Time = 0.40 (sec) , antiderivative size = 21, normalized size of antiderivative = 1.11

$$\int (a+b\csc(e+fx))^m\sin(e+fx)\,dx = \int (b\csc(fx+e)+a)^m\sin(fx+e)\,dx$$

input
```
integrate((a+b*csc(f*x+e))^m*sin(f*x+e),x, algorithm="giac")
```

output
```
integrate((b*csc(f*x + e) + a)^m*sin(f*x + e), x)
```

3.58.9 Mupad [N/A]

Not integrable

Time = 18.43 (sec) , antiderivative size = 23, normalized size of antiderivative = 1.21

$$\int (a+b\csc(e+fx))^m\sin(e+fx)\,dx = \int \sin(e+f\,x)\,\left(a+\frac{b}{\sin(e+f\,x)}\right)^m\,dx$$

input
```
int(sin(e + f*x)*(a + b/sin(e + f*x))^m,x)
```

output
```
int(sin(e + f*x)*(a + b/sin(e + f*x))^m, x)
```

3.59 $\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx$

3.59.1 Optimal result

Integrand size = 21, antiderivative size = 21

$$\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx = \operatorname{Int}\left((a+b\csc(e+fx))^m \sin^2(e+fx),x\right)$$

output
```
Unintegrable((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x)
```

3.59.2 Mathematica [N/A]

Not integrable

Time = 6.53 (sec) , antiderivative size = 23, normalized size of antiderivative = 1.10

$$\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx = \int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx$$

input
```
Integrate[(a + b*Csc[e + f*x])^m*Sin[e + f*x]^2,x]
```

output
```
Integrate[(a + b*Csc[e + f*x])^m*Sin[e + f*x]^2, x]
```

3.59.3 Rubi [N/A]

Not integrable

Time = 0.20 (sec) , antiderivative size = 21, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 0, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.000, Rules used = {3042, 4357}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sin^2(e+fx)(a+b\csc(e+fx))^m\,dx$$
$$\Big\downarrow \; 3042$$
$$\int \frac{(a+b\csc(e+fx))^m}{\csc(e+fx)^2}dx$$
$$\Big\downarrow \; 4357$$
$$\int \sin^2(e+fx)(a+b\csc(e+fx))^m dx$$

input
```
Int[(a + b*Csc[e + f*x])^m*Sin[e + f*x]^2,x]
```

output
```
$Aborted
```

3.59.3.1 Defintions of rubi rules used

rule 3042
```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4357
```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_))^(m_.), x_Symbol] :> Unintegrable[(d*Csc[e + f*x])^n*(a + b*Csc[e + f*
x])^m, x] /; FreeQ[{a, b, d, e, f, m, n}, x]
```

3.59.4 Maple [N/A] (verified)

Not integrable

Time = 1.04 (sec) , antiderivative size = 21, normalized size of antiderivative = 1.00

$$\int (a+b\csc (fx+e))^{m} \sin (fx+e)^{2} dx$$

input
```
int((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x)
```

output
```
int((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x)
```

3.59.5 Fricas [N/A]

Not integrable

Time = 0.26 (sec) , antiderivative size = 26, normalized size of antiderivative = 1.24

$$\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx = \int (b\csc (fx+e)+a)^{m} \sin (fx+e)^{2} \,dx$$

input
```
integrate((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="fricas")
```

output
```
integral(-(cos(f*x + e)^2 - 1)*(b*csc(f*x + e) + a)^m, x)
```

3.59.6 Sympy [N/A]

Not integrable

Time = 21.26 (sec) , antiderivative size = 20, normalized size of antiderivative = 0.95

$$\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx = \int (a+b\csc (e+fx))^{m} \sin^{2} (e+fx)\,dx$$

input
```
integrate((a+b*csc(f*x+e))**m*sin(f*x+e)**2,x)
```

output
```
Integral((a + b*csc(e + f*x))**m*sin(e + f*x)**2, x)
```

3.59.7 Maxima [N/A]

Not integrable

Time = 2.10 (sec) , antiderivative size = 23, normalized size of antiderivative = 1.10

$$\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx = \int (b\csc(fx+e)+a)^m \sin(fx+e)^2\,dx$$

input
```
integrate((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="maxima")
```

output
```
integrate((b*csc(f*x + e) + a)^m*sin(f*x + e)^2, x)
```

3.59.8 Giac [N/A]

Not integrable

Time = 0.40 (sec) , antiderivative size = 23, normalized size of antiderivative = 1.10

$$\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx = \int (b\csc(fx+e)+a)^m \sin(fx+e)^2\,dx$$

input
```
integrate((a+b*csc(f*x+e))^m*sin(f*x+e)^2,x, algorithm="giac")
```

output
```
integrate((b*csc(f*x + e) + a)^m*sin(f*x + e)^2, x)
```

3.59.9 Mupad [N/A]

Not integrable

Time = 19.49 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.19

$$\int (a+b\csc(e+fx))^m \sin^2(e+fx)\,dx = \int \sin(e+f\,x)^2\left(a+\frac{b}{\sin(e+f\,x)}\right)^m\,dx$$

input
```
int(sin(e + f*x)^2*(a + b/sin(e + f*x))^m,x)
```

output
```
int(sin(e + f*x)^2*(a + b/sin(e + f*x))^m, x)
```

CHAPTER 4

Appendix

4.1 Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

4.1.1 Mathematica and Rubi grading function

```
(* Original version thanks to Albert Rich emailed on 03/21/2017 *)
(* ::Package:: *)

(* Nasser: April 7,2022. add second output which gives reason for the grade *)
(*                       Small rewrite of logic in main function to make it*)
(*                       match Maple's logic. No change in functionality otherwise*)

(* ::Subsection:: *)
(*GradeAntiderivative[result,optimal]*)


(* ::Text:: *)
(*If result and optimal are mathematical expressions, *)
(*         GradeAntiderivative[result,optimal] returns*)
```

```
(*  "F" if the result fails to integrate an expression that*)
(*      is integrable*)
(*  "C" if result involves higher level functions than necessary*)
(*  "B" if result is more than twice the size of the optimal*)
(*      antiderivative*)
(*  "A" if result can be considered optimal*)


GradeAntiderivative[result_,optimal_] := Module[{expnResult,expnOptimal,leafCountResult,leafC
    expnResult  = ExpnType[result];
    expnOptimal = ExpnType[optimal];
    leafCountResult = LeafCount[result];
    leafCountOptimal = LeafCount[optimal];

    (*Print["expnResult=",expnResult," expnOptimal=",expnOptimal];*)
    If[expnResult<=expnOptimal,
        If[Not[FreeQ[result,Complex]], (*result contains complex*)
            If[Not[FreeQ[optimal,Complex]], (*optimal contains complex*)
                If[leafCountResult<=2*leafCountOptimal,
                    finalresult={"A"," "}
                    ,(*ELSE*)
                    finalresult={"B","Both result and optimal contain complex but leaf count
                  ]
            ,(*ELSE*)
                finalresult={"C","Result contains complex when optimal does not."}
            ]
        ,(*ELSE*)(*result does not contains complex*)
            If[leafCountResult<=2*leafCountOptimal,
               finalresult={"A"," "}
            ,(*ELSE*)
               finalresult={"B","Leaf count is larger than twice the leaf count of optimal. $
              ]
        ]
    ,(*ELSE*) (*expnResult>expnOptimal*)
        If[FreeQ[result,Integrate] && FreeQ[result,Int],
            finalresult={"C","Result contains higher order function than in optimal. Order "<
            ,
            finalresult={"F","Contains unresolved integral."}
        ]
    ];

    finalresult
]
```

```
(* ::Text:: *)
(*The following summarizes the type number assigned an *)
(*expression based on the functions it involves*)
(*1 = rational function*)
(*2 = algebraic function*)
(*3 = elementary function*)
(*4 = special function*)
(*5 = hyperpergeometric function*)
(*6 = appell function*)
(*7 = rootsum function*)
(*8 = integrate function*)
(*9 = unknown function*)


ExpnType[expn_] :=
  If[AtomQ[expn],
    1,
  If[ListQ[expn],
    Max[Map[ExpnType,expn]],
  If[Head[expn]===Power,
    If[IntegerQ[expn[[2]]],
      ExpnType[expn[[1]]],
    If[Head[expn[[2]]]===Rational,
      If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
        1,
      Max[ExpnType[expn[[1]]],2]],
    Max[ExpnType[expn[[1]]],ExpnType[expn[[2]]],3]]],
  If[Head[expn]===Plus || Head[expn]===Times,
    Max[ExpnType[First[expn]],ExpnType[Rest[expn]]],
  If[ElementaryFunctionQ[Head[expn]],
    Max[3,ExpnType[expn[[1]]]],
  If[SpecialFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],4]],
  If[HypergeometricFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],5]],
  If[AppellFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],6]],
  If[Head[expn]===RootSum,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
  If[Head[expn]===Integrate || Head[expn]===Int,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],8]],
  9]]]]]]]]]]
```

```
ElementaryFunctionQ[func_] :=
  MemberQ[{
  Exp,Log,
  Sin,Cos,Tan,Cot,Sec,Csc,
  ArcSin,ArcCos,ArcTan,ArcCot,ArcSec,ArcCsc,
  Sinh,Cosh,Tanh,Coth,Sech,Csch,
  ArcSinh,ArcCosh,ArcTanh,ArcCoth,ArcSech,ArcCsch
},func]


SpecialFunctionQ[func_] :=
  MemberQ[{
  Erf, Erfc, Erfi,
  FresnelS, FresnelC,
  ExpIntegralE, ExpIntegralEi, LogIntegral,
  SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
  Gamma, LogGamma, PolyGamma,
  Zeta, PolyLog, ProductLog,
  EllipticF, EllipticE, EllipticPi
},func]


HypergeometricFunctionQ[func_] :=
  MemberQ[{Hypergeometric1F1,Hypergeometric2F1,HypergeometricPFQ},func]


AppellFunctionQ[func_] :=
  MemberQ[{AppellF1},func]
```

4.1.2 Maple grading function

```
# File: GradeAntiderivative.mpl
# Original version thanks to Albert Rich emailed on 03/21/2017

#Nasser 03/22/2017  Use Maple leaf count instead since buildin
#Nasser 03/23/2017  missing 'ln' for ElementaryFunctionQ added
#Nasser 03/24/2017  corrected the check for complex result
#Nasser 10/27/2017  check for leafsize and do not call ExpnType()
#                   if leaf size is "too large". Set at 500,000
```

```
#Nasser 12/22/2019  Added debug flag, added 'dilog' to special functions
#                   see problem 156, file Apostol_Problems
#Nasser 4/07/2022   add second output which gives reason for the grade

GradeAntiderivative := proc(result,optimal)
local leaf_count_result,
        leaf_count_optimal,
        ExpnType_result,
        ExpnType_optimal,
        debug:=false;

        leaf_count_result:=leafcount(result);
        #do NOT call ExpnType() if leaf size is too large. Recursion problem
        if leaf_count_result > 500000 then
            return "B","result has leaf size over 500,000. Avoiding possible recursion issues
        fi;

        leaf_count_optimal := leafcount(optimal);
        ExpnType_result  := ExpnType(result);
        ExpnType_optimal := ExpnType(optimal);

        if debug then
            print("ExpnType_result",ExpnType_result," ExpnType_optimal=",ExpnType_optimal);
        fi;

# If result and optimal are mathematical expressions,
#  GradeAntiderivative[result,optimal] returns
#   "F" if the result fails to integrate an expression that
#       is integrable
#   "C" if result involves higher level functions than necessary
#   "B" if result is more than twice the size of the optimal
#       antiderivative
#   "A" if result can be considered optimal

    #This check below actually is not needed, since I only
    #call this grading only for passed integrals. i.e. I check
    #for "F" before calling this. But no harm of keeping it here.
    #just in case.


    if not type(result,freeof('int')) then
        return "F","Result contains unresolved integral";
    fi;
```

```
    if ExpnType_result<=ExpnType_optimal then
        if debug then
             print("ExpnType_result<=ExpnType_optimal");
        fi;
        if is_contains_complex(result) then
            if is_contains_complex(optimal) then
                if debug then
                        print("both result and optimal complex");
                fi;
                if leaf_count_result<=2*leaf_count_optimal then
                    return "A"," ";
                else
                   return "B",cat("Both result and optimal contain complex but leaf count of
                                   convert(leaf_count_result,string)," vs. $2 (",
                                   convert(leaf_count_optimal,string)," ) = ",convert(2*leaf_
                end if
            else #result contains complex but optimal is not
                if debug then
                        print("result contains complex but optimal is not");
                fi;
                return "C","Result contains complex when optimal does not.";
            fi;
        else # result do not contain complex
             # this assumes optimal do not as well. No check is needed here.
            if debug then
                   print("result do not contain complex, this assumes optimal do not as well"
            fi;
            if leaf_count_result<=2*leaf_count_optimal then
                if debug then
                    print("leaf_count_result<=2*leaf_count_optimal");
                fi;
                return "A"," ";
            else
                if debug then
                    print("leaf_count_result>2*leaf_count_optimal");
                fi;
                return "B",cat("Leaf count of result is larger than twice the leaf count of o
                                convert(leaf_count_result,string),"$ vs. $2(",
                                convert(leaf_count_optimal,string),")=",convert(2*leaf_cou
            fi;
        fi;
```

```
    else #ExpnType(result) > ExpnType(optimal)
        if debug then
            print("ExpnType(result) > ExpnType(optimal)");
        fi;
        return "C",cat("Result contains higher order function than in optimal. Order ",
                       convert(ExpnType_result,string)," vs. order ",
                       convert(ExpnType_optimal,string),".");
    fi;

end proc:

#
# is_contains_complex(result)
# takes expressions and returns true if it contains "I" else false
#
#Nasser 032417
is_contains_complex:= proc(expression)
  return (has(expression,I));
end proc:

# The following summarizes the type number assigned an expression
# based on the functions it involves
# 1 = rational function
# 2 = algebraic function
# 3 = elementary function
# 4 = special function
# 5 = hyperpergeometric function
# 6 = appell function
# 7 = rootsum function
# 8 = integrate function
# 9 = unknown function

ExpnType := proc(expn)
  if type(expn,'atomic') then
    1
  elif type(expn,'list') then
    apply(max,map(ExpnType,expn))
  elif type(expn,'sqrt') then
    if type(op(1,expn),'rational') then
       1
    else
       max(2,ExpnType(op(1,expn)))
    end if
```

```
  elif type(expn,'`^`') then
    if type(op(2,expn),'integer') then
      ExpnType(op(1,expn))
    elif type(op(2,expn),'rational') then
      if type(op(1,expn),'rational') then
         1
      else
         max(2,ExpnType(op(1,expn)))
      end if
    else
         max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    end if
  elif type(expn,'`+`') or type(expn,'`*`') then
    max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
  elif ElementaryFunctionQ(op(0,expn)) then
    max(3,ExpnType(op(1,expn)))
  elif SpecialFunctionQ(op(0,expn)) then
    max(4,apply(max,map(ExpnType,[op(expn)])))
  elif HypergeometricFunctionQ(op(0,expn)) then
    max(5,apply(max,map(ExpnType,[op(expn)])))
  elif AppellFunctionQ(op(0,expn)) then
    max(6,apply(max,map(ExpnType,[op(expn)])))
  elif op(0,expn)='int' then
    max(8,apply(max,map(ExpnType,[op(expn)]))) else
  9
  end if
end proc:


ElementaryFunctionQ := proc(func)
  member(func,[
        exp,log,ln,
        sin,cos,tan,cot,sec,csc,
        arcsin,arccos,arctan,arccot,arcsec,arccsc,
        sinh,cosh,tanh,coth,sech,csch,
        arcsinh,arccosh,arctanh,arccoth,arcsech,arccsch])
end proc:

SpecialFunctionQ := proc(func)
  member(func,[
        erf,erfc,erfi,
        FresnelS,FresnelC,
        Ei,Ei,Li,Si,Ci,Shi,Chi,
```

```
        GAMMA,lnGAMMA,Psi,Zeta,polylog,dilog,LambertW,
        EllipticF,EllipticE,EllipticPi])
end proc:

HypergeometricFunctionQ := proc(func)
  member(func,[Hypergeometric1F1,hypergeom,HypergeometricPFQ])
end proc:

AppellFunctionQ := proc(func)
  member(func,[AppellF1])
end proc:

# u is a sum or product.  rest(u) returns all but the
# first term or factor of u.
rest := proc(u) local v;
  if nops(u)=2 then
     op(2,u)
  else
     apply(op(0,u),op(2..nops(u),u))
  end if
end proc:

#leafcount(u) returns the number of nodes in u.
#Nasser 3/23/17  Replaced by build-in leafCount from package in Maple
leafcount := proc(u)
   MmaTranslator[Mma][LeafCount](u);
end proc:
```

4.1.3 Sympy grading function

```
#Dec 24, 2019. Nasser M. Abbasi:
#              Port of original Maple grading function by
#              Albert Rich to use with Sympy/Python
#Dec 27, 2019  Nasser. Added `RootSum`. See problem 177, Timofeev file
#              added 'exp_polar'
from sympy import *

def leaf_count(expr):
   #sympy do not have leaf count function. This is approximation
   return round(1.7*count_ops(expr))

def is_sqrt(expr):
```

```
    if isinstance(expr,Pow):
        if expr.args[1] == Rational(1,2):
            return True
        else:
            return False
    else:
        return False

def is_elementary_function(func):
    return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
            asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
            asinh,acosh,atanh,acoth,asech,acsch
        ]

def is_special_function(func):
    return func in [ erf,erfc,erfi,
             fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
             gamma,loggamma,digamma,zeta,polylog,LambertW,
             elliptic_f,elliptic_e,elliptic_pi,exp_polar
         ]

def is_hypergeometric_function(func):
    return func in [hyper]

def is_appell_function(func):
    return func in [appellf1]

def is_atom(expn):
    try:
        if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
            return True
        else:
            return False

    except AttributeError as error:
        return False

def expnType(expn):
    debug=False
    if debug:
        print("expn=",expn,"type(expn)=",type(expn))

    if is_atom(expn):
```

```
        return 1
    elif isinstance(expn,list):
        return max(map(expnType, expn))    #apply(max,map(ExpnType,expn))
    elif  is_sqrt(expn):
        if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
            return 1
        else:
            return max(2,expnType(expn.args[0]))  #max(2,ExpnType(op(1,expn)))
    elif isinstance(expn,Pow):    #type(expn,'`^`')
        if isinstance(expn.args[1],Integer):  #type(op(2,expn),'integer')
            return expnType(expn.args[0])   #ExpnType(op(1,expn))
        elif isinstance(expn.args[1],Rational):  #type(op(2,expn),'rational')
            if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
                return 1
            else:
                return max(2,expnType(expn.args[0]))  #max(2,ExpnType(op(1,expn)))
        else:
            return max(3,expnType(expn.args[0]),expnType(expn.args[1])) #max(3,ExpnType(op(1,expn)),ExpnT
    elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'`+`') or type(expn,'`*`')
        m1 = expnType(expn.args[0])
        m2 = expnType(list(expn.args[1:]))
        return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.func):  #ElementaryFunctionQ(op(0,expn))
        return max(3,expnType(expn.args[0]))  #max(3,ExpnType(op(1,expn)))
    elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(4,m1)    #max(4,apply(max,map(ExpnType,[op(expn)])))
    elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(5,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif is_appell_function(expn.func):
        m1 = max(map(expnType, list(expn.args)))
        return max(6,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif isinstance(expn,RootSum):
        m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
        return max(7,m1)
    elif str(expn).find("Integral") != -1:
        m1 = max(map(expnType, list(expn.args)))
        return max(8,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9

#main function
```

```
def grade_antiderivative(result,optimal):

    #print ("Enter grade_antiderivative for sagemath")
    #print("Enter grade_antiderivative, result=",result," optimal=",optimal)

    leaf_count_result  = leaf_count(result)
    leaf_count_optimal = leaf_count(optimal)

    #print("leaf_count_result=",leaf_count_result)
    #print("leaf_count_optimal=",leaf_count_optimal)

    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)

    if str(result).find("Integral") != -1:
        grade = "F"
        grade_annotation =""
    else:
        if expnType_result <= expnType_optimal:
            if result.has(I):
                if optimal.has(I): #both result and optimal complex
                    if leaf_count_result <= 2*leaf_count_optimal:
                        grade = "A"
                        grade_annotation =""
                    else:
                        grade = "B"
                        grade_annotation ="Both result and optimal contain complex but leaf count of result is larg
                else: #result contains complex but optimal is not
                    grade = "C"
                    grade_annotation ="Result contains complex when optimal does not."
            else: # result do not contain complex, this assumes optimal do not as well
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation =""
                else:
                    grade = "B"
                    grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(le
        else:
            grade = "C"
            grade_annotation ="Result contains higher order function than in optimal. Order "+str(ExpnType_


    #print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)
```

```
    return grade, grade_annotation
```

4.1.4 SageMath grading function

```
#Dec 24, 2019. Nasser: Ported original Maple grading function by
#              Albert Rich to use with Sagemath. This is used to
#              grade Fricas, Giac and Maxima results.
#Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
#              'arctan2','floor','abs','log_integral'
#June 4, 2022  Made default grade_annotation "none" instead of "" due
#              issue later when reading the file.
#July 14, 2022. Added ellipticF. This is until they fix sagemath, then remove it.

from sage.all import *
from sage.symbolic.operators import add_vararg, mul_vararg

debug=False;

def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    #print("Enter tree_size, expr is ",expr)

    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)

def is_sqrt(expr):
    if expr.operator() == operator.pow:   #isinstance(expr,Pow):
        if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
            if debug: print ("expr is sqrt")
            return True
        else:
```

```
        return False
    else:
        return False

def is_elementary_function(func):
    #debug=False
    m = func.name() in ['exp','log','ln',
            'sin','cos','tan','cot','sec','csc',
            'arcsin','arccos','arctan','arccot','arcsec','arccsc',
            'sinh','cosh','tanh','coth','sech','csch',
            'arcsinh','arccosh','arctanh','arccoth','arcsech','arccsch','sgn',
        'arctan2','floor','abs'
        ]
    if debug:
        if m:
            print ("func ", func , " is elementary_function")
        else:
            print ("func ", func , " is NOT elementary_function")


    return m

def is_special_function(func):
    #debug=False
    if debug:
        print ("type(func)=", type(func))

    m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
          'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','sinh_integral'
          'Chi','cosh_integral','gamma','log_gamma','psi,zeta',
          'polylog','lambert_w','elliptic_f','elliptic_e','ellipticF',
          'elliptic_pi','exp_integral_e','log_integral']

    if debug:
        print ("m=",m)
        if m:
            print ("func ", func ," is special_function")
        else:
            print ("func ", func ," is NOT special_function")


    return m
```

```
def is_hypergeometric_function(func):
   return func.name() in ['hypergeometric','hypergeometric_M','hypergeometric_U']

def is_appell_function(func):
   return func.name() in ['hypergeometric']   #[appellf1] can't find this in sagemath

def is_atom(expn):

   #debug=False
   if debug:
       print ("Enter is_atom, expn=",expn)

   if not hasattr(expn, 'parent'):
      return False


   #thanks to answer at https://ask.sagemath.org/question/49179/what-is-sagemath-equivalent-to-atomic-
   try:
      if expn.parent() is SR:
         return expn.operator() is None
      if expn.parent() in (ZZ, QQ, AA, QQbar):
         return expn in expn.parent() # Should always return True
      if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens"):
         return expn in expn.parent().base_ring() or expn in expn.parent().gens()

      return False

   except AttributeError as error:
      print("Exception,AttributeError in  is_atom")
      print ("cought exception" , type(error).__name__ )
      return False


def expnType(expn):

   if debug:
      print (">>>>>Enter expnType, expn=", expn)
      print (">>>>>is_atom(expn)=", is_atom(expn))

   if is_atom(expn):
      return 1
   elif type(expn)==list:   #isinstance(expn,list):
```

```
        return max(map(expnType, expn))    #apply(max,map(ExpnType,expn))
    elif  is_sqrt(expn):
        if  type(expn.operands()[0])==Rational: #type(isinstance(expn.args[0],Rational):
            return 1
        else:
            return max(2,expnType(expn.operands()[0]))   #max(2,expnType(expn.args[0]))
    elif expn.operator() == operator.pow:   #isinstance(expn,Pow)
        if type(expn.operands()[1])==Integer:  #isinstance(expn.args[1],Integer)
            return expnType(expn.operands()[0])   #expnType(expn.args[0])
        elif type(expn.operands()[1])==Rational:  #isinstance(expn.args[1],Rational)
            if type(expn.operands()[0])==Rational: #isinstance(expn.args[0],Rational)
                return 1
            else:
                return max(2,expnType(expn.operands()[0]))  #max(2,expnType(expn.args[0]))
        else:
            return max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1])) #max(3,expnType(expn.
    elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #isinstance(expn,Add) or isinst
        m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
        m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
        return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.operator()):  #is_elementary_function(expn.func)
        return max(3,expnType(expn.operands()[0]))
    elif is_special_function(expn.operator()): #is_special_function(expn.func)
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(4,m1)   #max(4,m1)
    elif is_hypergeometric_function(expn.operator()): #is_hypergeometric_function(expn.func)
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(5,m1)   #max(5,m1)
    elif is_appell_function(expn.operator()):
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(6,m1)   #max(6,m1)
    elif str(expn).find("Integral") != -1: #this will never happen, since it
                #is checked before calling the grading function that is passed.
                #but kept it here.
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(8,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9

#main function
def grade_antiderivative(result,optimal):
```

```
    if debug:
        print ("Enter grade_antiderivative for sagemath")
        print("Enter grade_antiderivative, result=",result)
        print("Enter grade_antiderivative, optimal=",optimal)
        print("type(anti)=",type(result))
        print("type(optimal)=",type(optimal))

    leaf_count_result  = tree_size(result) #leaf_count(result)
    leaf_count_optimal = tree_size(optimal) #leaf_count(optimal)

    #if debug: print ("leaf_count_result=", leaf_count_result, "leaf_count_optimal=",leaf_count_optimal)


    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)

    if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",expnType_optimal)

    if expnType_result <= expnType_optimal:
        if result.has(I):
            if optimal.has(I): #both result and optimal complex
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation =" "
                else:
                    grade = "B"
                    grade_annotation ="Both result and optimal contain complex but leaf count of result is larger t
            else: #result contains complex but optimal is not
                grade = "C"
                grade_annotation ="Result contains complex when optimal does not."
        else: # result do not contain complex, this assumes optimal do not as well
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation =" "
            else:
                grade = "B"
                grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(leaf_
    else:
        grade = "C"
        grade_annotation ="Result contains higher order function than in optimal. Order "+str(expnType_resu


    print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)
```

```
    return grade, grade_annotation
```